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ON A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS WITH MISSING COEFFICIENTS

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Abstract. The purpose of the present paper is to introduce a new subclass of harmonic univalent functions defined by convolution. Coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combinations are studied for this class. Finally, we discuss a class preserving integral operator for this class.

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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected domain D is said to be harmonic in D if both u and v are harmonic in D. In any simply-connected domain D we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$. See Clunie and Sheil-Small [3]. For more basic results on harmonic mappings one may refer to the following excellent text book by Duren [5], (see also Ahuja [1], Ponnusamy and Rasila [8], [9] and references there in).

Denote by S_H^j the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=j+1}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \quad (1)$$

The harmonic function $f = h + \bar{g}$ for $g \equiv 0$ reduces to an analytic function $f \equiv h$. A function $f = h + \bar{g}$ of the form (1) is said to be harmonic starlike of order

$\alpha(0 \leq \alpha < 1)$ in U, if and only if

$$\frac{\partial}{\partial\theta}\{\arg f(re^{i\theta})\} = \Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > \alpha, z \in U. \tag{2}$$

The classes of all harmonic starlike functions of order α is denoted by $S_H^{j,*}(\alpha)$. For $j = 1$, this class have been extensively studied by Jahangiri [6].

For $\alpha = 0; j = 1$ the class $S_H^{j,*}(\alpha)$ is denoted by S_H^* and studied in detail by Silverman [12] and Silverman and Silvia [13], (see also [2]).

Let TS_H^j denote the class consisting of functions of the form

$$h(z) = z - \sum_{k=j+1}^{\infty} |a_k|z^k, g(z) = \sum_{k=1}^{\infty} |b_k|z^k, |b_1| < 1. \tag{3}$$

Let $HP^j(\varphi, \Psi, \beta)$ denote the subclass of S_H^j satisfying the condition

$$\Re\left\{\frac{h(z) * \varphi(z) + g(z) * \Psi(z)}{z}\right\} > \beta, 0 \leqslant \beta < 1, \tag{4}$$

where $\varphi(z) = z + \sum_{k=j+1}^{\infty} \lambda_k z^k$ and $\Psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k$ are analytic in U with the conditions $\lambda_k \geq 0; \mu_k \geq 0$. The operator " $*$ " stands for the Hadamard product or convolution of two power series. We further denote by $THP^j(\varphi, \Psi, \beta)$ the subclass of $HP^j(\varphi, \Psi, \beta)$ such that the functions h and g in $f = h + \bar{g}$ are of the form (3).

Clearly, if $0 \leqslant \beta_1 \leqslant \beta_2 < 1$, then $HP^j(\varphi, \Psi; \beta_2) \subseteq HP^j(\varphi, \Psi; \beta_1)$.

We note that for $j = 1$, the class $HP^j(\varphi, \Psi, \beta)$ reduces to the class $HP(\varphi, \Psi, \beta)$ studied by Porwal *et al.* [11], (see also [10]) and $HP^1\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \beta\right) = HP(\beta)$ and $THP^1\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \beta\right) = HP^*(\beta)$ were studied by Karpuzogullari *et al.* [7], (see also [4]).

In the present paper, results involving the coefficient inequalities, extreme points, distortion bounds, convolution condition and convex combinations for the above classes $HP^j(\varphi, \Psi, \beta)$ and $THP^j(\varphi, \Psi, \beta)$ of harmonic univalent functions have been investigated.

2. Main Results

First, we give a sufficient coefficient condition for function $f = h + \overline{g} \in S_H^j$ belonging to the class $HP^j(\varphi, \Psi, \beta)$.

Theorem 1. *Let the function* $f = h + \overline{g}$ *be such that* h *and* g *are given by* (1). *Furthermore, let*

$$\sum_{k=j+1}^{\infty} \lambda_k |a_k| + \sum_{k=1}^{\infty} \mu_k |b_k| \leqslant 1 - \beta, \tag{5}$$

where $0 \leqslant \beta < 1, k(1-\beta) \leqslant \lambda_k, k(1-\beta) \leqslant \mu_k$. *Then* f *is sense-preserving, harmonic univalent in* U *and* $f \in HP^j(\varphi, \Psi, \beta)$.

Proof. First we note that f is locally univalent and sense-preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geqslant 1 - \sum_{k=j+1}^{\infty} k|a_k| r^{k-1} \\
&> \sum_{k=j+1}^{\infty} k|a_k| \\
&\geq 1 - \sum_{k=j+1}^{\infty} \frac{\lambda_k}{1-\beta} |a_k| \\
&\geqslant \sum_{k=1}^{\infty} \frac{\mu_k}{1-\beta} |b_k| \\
&\geqslant \sum_{k=1}^{\infty} k|b_k| \\
&> \sum_{k=1}^{\infty} k|b_k| r^{k-1} \\
&\geqslant |g'(z)|.
\end{aligned}$$

To show that f is univalent in U, suppose $z_1, z_2 \in U$ such that $z_1 \neq z_2$. Then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geqslant 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1-\left|\frac{\sum\limits_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{z_1-z_2+\sum\limits_{k=j+1}^{\infty} a_k(z_1^k-z_2^k)}\right| \\
&> 1-\frac{\sum\limits_{k=1}^{\infty} k|b_k|}{1-\sum\limits_{k=j+1}^{\infty} k|a_k|} \\
&\geqslant 1-\frac{\sum\limits_{k=1}^{\infty} \frac{\mu_k}{1-\beta}|b_k|}{1-\sum\limits_{k=j+1}^{\infty} \frac{\lambda_k}{1-\beta}|a_k|} \\
&\geq 0.
\end{aligned}$$

Now, we show that $f \in HP^j(\varphi, \Psi; \beta)$. Using the fact that $\Re\{\omega\} \geq \beta$, if and only if $|1-\beta+w| \geqslant |1+\beta-w|$, it suffices to show that

$$\left|1-\beta+\frac{h(z)*\varphi(z)+g(z)*\Psi(z)}{z}\right|-\left|1+\beta-\frac{h(z)*\varphi(z)+g(z)*\Psi(z)}{z}\right| \geqslant 0. \quad (6)$$

Substituting the values of $h(z)*\varphi(z)$ and $g(z)*\Psi(z)$ in L.H.S. of (6), we have

$$\begin{aligned}
&\left|(1-\beta)+1+\sum_{k=j+1}^{\infty} \lambda_k a_k z^{k-1}+\sum_{k=1}^{\infty} \mu_k b_k z^{k-1}\right| \\
&-\left|(1+\beta)-1-\sum_{k=j+1}^{\infty} \lambda_k a_k z^{k-1}-\sum_{k=1}^{\infty} \mu_k b_k z^{k-1}\right| \\
&=\left|2-\beta+\sum_{k=j+1}^{\infty} \lambda_k a_k z^{k-1}+\sum_{k=1}^{\infty} \mu_k b_k z^{k-1}\right| \\
&-\left|\beta-\sum_{k=j+1}^{\infty} \lambda_k a_k z^{k-1}-\sum_{k=1}^{\infty} \mu_k b_k z^{k-1}\right|
\end{aligned}$$

$$\geqslant (2-\beta) - \sum_{k=j+1}^{\infty} \lambda_k |a_k||z|^{k-1} - \sum_{k=1}^{\infty} \mu_k |b_k||z|^{k-1}$$

$$-\beta - \sum_{k=j+1}^{\infty} \lambda_k |a_k||z|^{k-1} - \sum_{k=1}^{\infty} \mu_k |b_k||z|^{k-1}$$

$$> 2\left[(1-\beta) - \sum_{k=j+1}^{\infty} \lambda_k |a_k| - \sum_{k=1}^{\infty} \mu_k |b_k|\right]$$

$$\geq 0, \qquad \text{by(5)}.$$

The harmonic mappings

$$f(z) = z + \sum_{k=j+1}^{\infty} \frac{1-\beta}{\lambda_k} x_k z^k + \sum_{k=1}^{\infty} \frac{1-\beta}{\mu_k} \overline{y_k z^k}, \tag{7}$$

where $\sum_{k=j+1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (5) is sharp.

In our next theorem, we prove that the above sufficient condition is also necessary for functions in $THP^j(\varphi, \Psi, \beta)$.

Theorem 2. *Let $f = h + \overline{g}$ be given by* (3)*. Then $f \in THP^j(\varphi, \Psi, \beta)$, if and only if*

$$\sum_{k=j+1}^{\infty} \lambda_k |a_k| + \sum_{k=1}^{\infty} \mu_k |b_k| \leqslant 1-\beta, \tag{8}$$

where $0 \leqslant \beta < 1, k(1-\beta) \leqslant \lambda_k$, $(k \geqslant j+1, j+2, ...)$ and $k(1-\beta) \leqslant \mu_k$, for $k \geq 1$.

Proof. Since $THP^j(\varphi, \Psi; \beta) \subseteq HP^j(\varphi, \Psi; \beta)$, we only need to prove the "only if" part of the theorem. For this we have to show that if $f \in THP^j(\varphi, \psi, \beta)$ then the condition (8) holds. We note that a necessary and sufficient condition for $f = h + \overline{g}$, given by (3), to be in the class $THP^j(\varphi, \Psi; \beta)$ is

$$\Re\left\{\frac{\varphi * h(z) + \Psi(z) * g(z)}{z}\right\} > \beta,$$

which is equivalent to

$$\Re\left\{1 - \sum_{k=j+1}^{\infty} \lambda_k |a_k| z^{k-1} - \sum_{k=1}^{\infty} \mu_k |b_k| z^{k-1}\right\} > \beta.$$

If we choose z to be real and let $z \to 1^{-1}$, we obtain

$$1-\sum_{k=j+1}^{\infty}\lambda_k|a_k|-\sum_{k=1}^{\infty}\mu_k|b_k| \geqslant \beta,$$

or

$$\sum_{k=j+1}^{\infty}\lambda_k|a_k|+\sum_{k=1}^{\infty}\mu_k|b_k| \leqslant 1-\beta,$$

which is the required condition.

Next, we give the bounds for the function belonging to the class $THP^j(\varphi,\Psi,\beta)$.

Theorem 3. *Let $f \in THP^j(\varphi,\Psi;\beta), A_{j+1} \leqslant \lambda_k$ and $A_{j+1} \leqslant \mu_k$ for $k \geq j+1$ and $A_{j+1} = min_k\{\lambda_k,\mu_k\}$. Then we have*

$$|f(z)| \leqslant (1+|b_1|)r+|b_2|\,r^2+...+|b_j|\,r^j+\frac{1}{A_{j+1}}(1-\beta-|b_1|-\mu_2\,|b_2|-...-\mu_j\,|b_j|)r^{j+1}, |z|=r<1$$

and

$$|f(z)| \geqslant (1-|b_1|)r-|b_2|\,r^2-...-|b_j|\,r^j-\frac{1}{A_{j+1}}(1-\beta-|b_1|-\mu_2\,|b_2|-...-\mu_j\,|b_j|)r^{j+1}, |z|=r<1$$

Proof. Let $f \in THP^j(\varphi,\Psi;\beta)$. Then we have

$$\begin{aligned}|f(z)| &\leqslant (1+|b_1|)r+\sum_{k=j+1}^{\infty}|a_k|r^k+\sum_{k=2}^{\infty}|b_k|r^k\\ &\leqslant (1+|b_1|)r+|b_2|\,r^2+...+|b_j|\,r^j+\sum_{k=j+1}^{\infty}(|a_k|+|b_k|)r^{j+1}\\ &= (1+|b_1|)r+|b_2|\,r^2+...+|b_j|\,r^j+\frac{1}{A_{j+1}}\sum_{k=j+1}^{\infty}A_{j+1}(|a_k|+|b_k|)r^{j+1}\\ &\leqslant (1+|b_1|)r+|b_2|\,r^2+...+|b_j|\,r^j+\frac{1}{A_{j+1}}\sum_{k=j+1}^{\infty}(\lambda_k|a_k|+\mu_k|b_k|)r^{j+1}\\ &\leqslant (1+|b_1|)r+|b_2|\,r^2+...+|b_j|\,r^j+\frac{1}{A_{j+1}}(1-\beta-|b_1|-\mu_2\,|b_2|-...-\mu_j\,|b_j|)r^{j+1}\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geqslant (1-|b_1|)r - \sum_{k=j+1}^{\infty} |a_k| r^k - \sum_{k=2}^{\infty} |b_k| r^k \\
&\geqslant (1-|b_1|)r - |b_2|\, r^2 - \dots - |b_j|\, r^j - \sum_{k=j+1}^{\infty} (|a_k|+|b_k|) r^{j+1} \\
&= (1-|b_1|)r - |b_2|\, r^2 - \dots - |b_j|\, r^j - \frac{1}{A_{j+1}} \sum_{k=j+1}^{\infty} A_{j+1}(|a_k|+|b_k|) r^{j+1} \\
&\geqslant (1-|b_1|)r - |b_2|\, r^2 - \dots - |b_j|\, r^j - \frac{1}{A_{j+1}} \sum_{k=j+1}^{\infty} (\lambda_k|a_k|+\mu_k|b_k|) r^{j+1} \\
&\geqslant (1-|b_1|)r - |b_2|\, r^2 - \dots - |b_j|\, r^j - \frac{1}{A_{j+1}} (1-\beta-|b_1|-\mu_2\,|b_2|-\dots-\mu_j\,|b_j|) r^{j+1}.
\end{aligned}$$

Next, we determine the extreme points of the closed convex hulls of $THP^j(\varphi,\Psi;\beta)$ denoted by clco $THP^j(\varphi,\Psi;\beta)$.

Theorem 4. *$f \in clcoTHP^j(\varphi,\Psi;\beta)$, if and only if*

$$f(z) = x_1 h_1(z) + \sum_{k=j+1}^{\infty} x_k h_k(z) + \sum_{k=1}^{\infty} y_k g_k(z), \tag{9}$$

where

$$h_1(z) = z, h_k(z) = z - \frac{1-\beta}{\lambda_k} z^k, k = j+1, j+2, \dots,$$

$$g_k(z) = z - \frac{1-\beta}{\mu_k} \overline{z}^k, k = 1, 2, \dots$$

and

$$\sum_{k=j+1}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1, x_k \geqslant 0 and y_k \geqslant 0.$$

In particular, the extreme points of $THP^j(\varphi,\Psi;\beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (9), we have

$$f(z) = x_1 h_1(z) + \sum_{k=j+1}^{\infty} x_k h_k(z) + \sum_{k=1}^{\infty} y_k g_k(z),$$

$$= z - \sum_{k=j+1}^{\infty} \frac{1-\beta}{\lambda_k} x_k z^k - \sum_{k=1}^{\infty} \frac{1-\beta}{\mu_k} y_k \overline{z}^k.$$

Then

$$\sum_{k=j+1}^{\infty} \frac{\lambda_k}{1-\beta}\left(\frac{1-\beta}{\lambda_k} x_k\right) + \sum_{k=1}^{\infty} \frac{\mu_k}{1-\beta}\left(\frac{1-\beta}{\mu_k} y_k\right)$$

$$= \sum_{k=j+1}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leqslant 1$$

and so $f \in \mathrm{clco}THP^j(\varphi, \Psi; \beta)$.

Conversely, suppose that $f \in \mathrm{clco}THP^j(\varphi, \Psi; \beta)$. Set

$$x_k = \frac{\lambda_k}{1-\beta}|a_k|,\ k = j+1, j+2, \ldots \mathrm{and} y_k = \frac{\mu_k}{1-\beta}|b_k|,\ k = 1, 2, 3, \ldots.$$

Then note that by Theorem 2, $0 \leqslant x_k \leqslant 1, (k = j+1, j+2, \ldots)$ and $0 \leqslant y_k \leqslant 1, (k = 1, 2, 3, \ldots)$, we define

$$x_1 = 1 - \sum_{k=j+1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$$

and note that, by Theorem 2, $x_1 \geq 0$. Consequently, we obtain

$$f(z) = x_1 h_1(z) + \sum_{k=j+1}^{\infty} x_k h_k(z) + \sum_{k=1}^{\infty} y_k g_k(z),$$

as required.

Theorem 5. *Each member of* $THP^j(\varphi, \Psi; \beta)$ *maps* U *onto a starlike domain.*

Proof. We only need to show that if $f = h + \overline{g} \in THP^j(\varphi, \Psi; \beta)$, then

$$\Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0.$$

Using the fact that $\Re\{\omega\} > 0$ if and only if $|1 + w| > |1 - w|$, it suffices to show that

$$|h(z) + \overline{g(z)} + zh'(z) - \overline{zg'(z)}| - |h(z) + \overline{g(z)} - zh'(z) + \overline{zg'(z)}|$$

$$
\begin{aligned}
&= \left| 2z - \sum_{k=j+1}^{\infty} (k+1)|a_k|z^k + \sum_{k=1}^{\infty}(k-1)|b_k|\overline{z}^k \right| \\
&- \left| \sum_{k=j+1}^{\infty} (k-1)|a_k|z^k + \sum_{k=1}^{\infty}(k+1)|b_k|\overline{z}^k \right| \\
&\geqslant 2|z| \left[1 - \sum_{k=j+1}^{\infty} k|a_k||z|^{k-1} - \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \right] \\
&> 2|z| \left[1 - \sum_{k=j+1}^{\infty} k|a_k| - \sum_{k=1}^{\infty} k|b_k| \right] \\
&\geqslant 2|z| \left[1 - \sum_{k=j+1}^{\infty} \frac{\lambda_k}{1-\beta}|a_k| - \sum_{k=1}^{\infty} \frac{\mu_k}{1-\beta}|b_k| \right] \\
&\geqslant 0.
\end{aligned}
$$

This completes the proof of theorem.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} |a_k|z^k - \sum_{k=1}^{\infty} |b_k|\overline{z}^k$$

and

$$F(z) = z - \sum_{k=j+1}^{\infty} |A_k|z^k - \sum_{k=1}^{\infty} |B_k|\overline{z}^k$$

we define their convolution

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=j+1}^{\infty} |a_k A_k|z^k - \sum_{k=1}^{\infty} |b_k B_k|\overline{z}^k. \tag{10}$$

Using this definition, we show that the class $THP^j(\varphi, \Psi; \beta)$ is closed under convolution.

Theorem 6. *For $0 \leqslant \alpha \leqslant \beta < 1$, let $f(z) \in THP^j(\varphi, \psi; \beta)$ and $F(z) \in THP^j(\varphi, \psi; \alpha)$. Then*

$$(f * F)(z) \in THP^j(\varphi, \psi; \beta) \subseteq THP^j(\varphi, \psi; \alpha).$$

Proof. Let $f(z) = z - \sum_{k=j+1}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k$ be in $THP^j(\varphi, \psi; \beta)$ and $F(z) = z - \sum\limits_{k=j+1}^{\infty} |A_k| z^k - \sum\limits_{k=1}^{\infty} |B_k| \overline{z}^k$ be in $THP^j(\varphi, \psi; \alpha)$. Then the convolution $(f * F)(z)$ is given by (10). We wish to show that the coefficient of $(f * F)(z)$ satisfy the required condition given in Theorem 2. For $F(z) \in THP^j(\varphi, \psi; \alpha)$, we note that $|A_k| \leqslant 1, (k = j+1, j+2, \ldots)$ and $|B_k| \leqslant 1, (k = 1, 2, 3, \ldots)$. Now, for the convolution functions $(f * F)(z)$, we have

$$\begin{aligned} &\sum_{k=j+1}^{\infty} \frac{\lambda_k}{1-\beta} |a_k A_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{1-\beta} |b_k B_k| \\ &\leqslant \sum_{k=j+1}^{\infty} \frac{\lambda_k}{1-\beta} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{1-\beta} |b_k| \\ &\leqslant 1, \text{since} f \in THP^j(\varphi, \Psi; \beta). \end{aligned}$$

Next, we show that the class $THP^j(\varphi, \Psi; \beta)$ is closed under convex combination.

Theorem 7. *The class $THP^j(\varphi, \Psi; \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3...$, let$f_i\,(z) \in THP^j(\varphi, \Psi; \beta)$, where $f_i(z)$ is given by

$$f_i(z) = z - \sum_{k=j+1}^{\infty} |a_{k_i}| z^k - \sum_{k=1}^{\infty} |b_{k_i}| \overline{z}^k.$$

Then by Theorem 2, we have

$$\sum_{k=j+1}^{\infty} \frac{\lambda_k}{1-\beta} |a_{k_i}| + \sum_{k=1}^{\infty} \frac{\mu_k}{1-\beta} |b_{k_i}| \leqslant 1. \tag{11}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=j+1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k - \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \overline{z}^k.$$

Then by (8), we have

$$\begin{aligned}
&\sum_{k=j+1}^{\infty}\frac{\lambda_k}{1-\beta}\left\{\sum_{i=1}^{\infty}t_i|a_{k_i}|\right\}+\sum_{k=1}^{\infty}\frac{\mu_k}{1-\beta}\left\{\sum_{i=1}^{\infty}t_i|b_{k_i}|\right\}\\
&==\sum_{i=1}^{\infty}t_i\left\{\sum_{k=j+1}^{\infty}\frac{\lambda_k}{1-\beta}|a_{k_i}|+\sum_{k=1}^{\infty}\frac{\mu_k}{1-\beta}|b_{k_i}|\right\}\\
&\le\sum_{i=1}^{\infty}t_i\\
&=1.
\end{aligned}$$

This is the condition required by Theorem 2 and so $\sum_{i=1}^{\infty}t_if_i(z)\in THP^j(\varphi,\Psi;\beta)$.

3. A Family of Class Preserving Integral Operator

Let $f(z)=h(z)+\overline{g(z)}$ be defined by (1), then $F(z)$ defined by the relation

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)\,dt+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)\,dt},(c>-1). \tag{12}$$

Theorem 8. *Let $f(z)=h(z)+\overline{g(z)}\in S_H$ be given by (3) and $f\in THP^j(\varphi,\Psi;\beta)$ then $F(z)$ be defined by (12) also belong to $THP^j(\varphi,\Psi;\beta)$.*

Proof. From the representation (12) of $F(z)$, is follows that

$$F(z)=z-\sum_{k=j+1}^{\infty}\frac{c+1}{c+k}|a_k|z^k-\sum_{k=1}^{\infty}\frac{c+1}{c+k}|b_k|\overline{z}^k.$$

Since $f(z)\in THP^j(\varphi,\Psi;\beta)$, we have

$$\sum_{k=j+1}^{\infty}\frac{\lambda_k}{1-\beta}|a_k|+\sum_{k=1}^{\infty}\frac{\mu_k}{1-\beta}|b_k|\leqslant 1. \tag{13}$$

Now

$$\sum_{k=j+1}^{\infty}\frac{\lambda_k}{1-\beta}\left(\frac{c+1}{c+k}|a_k|\right)+\sum_{k=1}^{\infty}\frac{\mu_k}{1-\beta}\left(\frac{c+1}{c+k}|b_k|\right)$$

$$\leqslant \sum_{k=j+1}^{\infty} \frac{\lambda_k}{1-\beta}|a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{1-\beta}|b_k|$$
$$\leqslant 1.$$

Thus $F(z) \in THP^j(\varphi, \Psi; \beta)$.

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