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A CLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY $Q-$ANALOGUE LINEAR OPERATOR

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Abstract. In the present investigation, we define a class of meromorphic functions by making use of the $q-$analogue of a linear operator. Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. We also establish some results concerning the partial sums of meromorphic functions in this class.

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1. Introduction

Let Φ denote the class of functions of the form:

$$\mathcal{F}(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k, \tag{1}$$

which are regular in $\mathbb{U}^*=\{z:0<|z|<1\}$. Also let Φ_δ denote the subclass of Φ consisting of functions of the form:

$$\mathcal{F}(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k,\ (a_k\geq 0), \tag{2}$$

which are analytic and univalent in $\mathbb{U}^*$.

For $0\leq\alpha<1$, the function $\mathcal{F}\in\Phi_\delta$ is said to be meromorphically starlike of order α and meromorphically convex of order α, respectively, if and only if

$$-Re\left\{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right\}>\alpha, \tag{3}$$

$$-\, Re\left\{1+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}\right\}>\alpha \tag{4}$$

The classes of such functions are denoted by $\Phi_\delta^*(\alpha)$ and $\Phi_\delta^c(\alpha)$, respectively. Note that the class $\Phi_\delta^*(\alpha)$ and various other subclasses of $\Phi_\delta^*(0)$ have been studied by [9], [12, 13, 14, 15] (see also [3], [7], [17], [19, 20, 21]). Aldweby and Darus [1] defined the basic hypergeometric function ${}_lF_s(a_1,...,a_l;b_1,...,b_s,q,z)$, for complex parameters a_i, b_j, q $(i=1,...,l,\ j=1,...,s,\ b_j\in\mathbb{C}\setminus\{0,-1,-2,...\},\ |q|<1)$, by

$${}_lF_s(a_1,...,a_l;b_1,...,b_s;q;z)=\sum_{k=0}^{\infty}\frac{(a_1,q)_k...(a_l,q)_k}{(q,q)_k(b_1,q)_k...(b_s,q)_k}z^k, \tag{5}$$

$(l\le s+1,\ l,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ z\in\mathbb{U}^*)$ where $\mathbb{N}$ denotes the set of positive integers and $(a,q)_k$ is the $q-$shifted factorial defined by

$$(a,q)_k=\begin{cases}1, & k=0;\\ (1-a)(1-aq)(1-aq^2)...(1-aq^{k-1}), & k\in\mathbb{N}; a\in\mathbb{C}\end{cases}. \tag{6}$$

We note that

$$\begin{aligned}&\lim_{q\to1^-}\left[{}_lF_s(q^{a_1},...,q^{a_l};q^{b_1},...,q^{b_s};q;(q-1)^{1+s-l}z)\right]\\ =\ &{}_lF_s(a_1,...,a_l;b_1,...,b_s;z),\end{aligned} \tag{7}$$

the well-known generalized hypergeometric function. For more mathematical background of basic hypergeometric functions, one may refer to [5, 6].

It is known that the calculus without the notion of limits is called $q-$calculus which has influenced many scientific fields due to its important applications. Tang et al. [18] defined the $q-$derivative $\partial_q(\mathcal{F}(z))$ by:

$$\begin{aligned}\partial_q\mathcal{F}(z) &= \frac{\mathcal{F}(z)-\mathcal{F}(qz)}{(1-q)z}\\ &= -\frac{1}{qz^2}+\sum_{k=1}^{\infty}[k]_q a_k z^{k-1},\end{aligned} \tag{8}$$

where

$$[j]_q=\frac{1-q^j}{1-q}. \tag{9}$$

As $q\to1^-$, $[j]_q=j$ and $\partial_q\mathcal{F}(z)=\mathcal{F}'(z)$.

For positive real values of $a_1,...,a_l$ and $b_1,...,b_s$ $(b_j \in \mathbb{C}\setminus\{0,-1,-2,...\}, j=1,...,s)$, let

$$\mathcal{H}(a_1,...,a_l;b_1,...,b_s;q):\Phi\to\Phi$$

be a linear operator defined by

$$\begin{aligned}\mathcal{H}(a_1,...,a_l;b_1,...,b_s;q,z) &= \mathcal{H}_{l,s,q}(a_1)=z^{-1}\ {}_lF_s(a_1,...,a_l;b_1,...,b_s;q;z)\\ &= z^{-1}+\sum_{k=1}^{\infty}\Gamma_{q,k}z^k, \end{aligned}\tag{10}$$

where

$$\Gamma_{q,k}=\frac{(a_1,q)_{k+1}...(a_l,q)_{k+1}}{(q,q)_{k+1}(b_1,q)_{k+1}...(b_s,q)_{k+1}}. \tag{11}$$

Note that $\lim_{q\to1-}\mathcal{H}_{l,s,q}(a_1)=\mathcal{H}_{l,s}(a_1)$ was investigated recently by Liu and Srivastava [8] and Aouf [2]. With the aid of the function $\mathcal{H}_{l,s,q}$, let

$$\mathcal{H}_{l,s,q}*\mathcal{H}^*_{l,s,q}=\mathcal{G}_{q,\lambda+1}(z),\ (z\in\mathbb{U}^*;\lambda>-1). \tag{12}$$

where

$$\mathcal{G}_{q,\lambda+1}(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{[\lambda+1,q]_{k+1}}{[k+1,q]!}z^k, \tag{13}$$

and $[k+1,q]!=\begin{cases}1, & if\ k=0\\ [1,q][2,q][3,q]...[k,q][k+1,q], & if\ k\in\mathbb{N}\end{cases}$.

This function yields the following family of linear operators $\mathcal{M}^{\lambda}_{l,s,q}:\Phi\to\Phi$ which are given by:

$$\mathcal{M}^{\lambda}_{l,s,q}(a_1)\mathcal{F}(z)=\mathcal{H}^*_{l,s,q}*\mathcal{F}(z). \tag{14}$$

If $\mathcal{F}(z)$ is given by (2), then

$$\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z)=\mathcal{M}^{\lambda}_{l,s,q}(a_1)\mathcal{F}(z)=z^{-1}+\sum_{k=0}^{\infty}\Gamma_{q,k}(\lambda)a_kz^k,\ (z\in\mathbb{U}^*,\lambda>-1). \tag{15}$$

where

$$\Gamma_{q,k}(\lambda)=\frac{(q,q)_{k+1}(b_1,q)_{k+1}...(b_s,q)_{k+1}[\lambda+1,q]_{k+1}}{(a_1,q)_{k+1}...(a_l,q)_{k+1}[k+1,q]!}. \tag{16}$$

Note that: $\lim_{q\to1^-}\mathcal{M}^{\lambda}_{l,s,q}(a_1)\mathcal{F}(z)=\mathcal{M}^{\lambda}_{l,s}(a_1)\mathcal{F}(z)$ (see [10] at $p=1$).

Definition 1. *The function $\mathcal{F}\in\Phi_\delta$ is said to be in the class $\Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$ if it satisfies*

$$Re\left\{\frac{zq\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z))}{(\zeta-1)\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z)+q\zeta z\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z))}\right\}>\alpha, \tag{17}$$

where $\lambda>-1$, $0\le\alpha<1$, $0\le\zeta<1$.

2. Main Results

Unless indicated, let $0 < q < 1$, $0 \leq \alpha < 1$, $0 \leq \zeta < 1$, $\lambda > -1$, $z \in \mathbb{U}^*$, $\mathcal{F}(z)$ defined by (2).

Theorem 1. *The function $\mathcal{F} \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} [q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\, \Gamma_{q,k}(\lambda) a_k \leq 1-\alpha. \tag{18}$$

Proof. Assume that (18) holds true. Since

$$Re\{\omega\} > \alpha \quad \text{if and only if} \quad |\omega - 1| < |\omega + 1 - 2\alpha|,$$

it is sufficient to show that

$$\left| \frac{zq\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z)) - \left[(\zeta-1)\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z) + q\zeta z\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z))\right]}{zq\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z)) + (1-2\alpha)\left[(\zeta-1)\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z) + q\zeta z\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z))\right]} \right| < 1.$$

Using (18), we have for $0 < |z| = r < 1$,

$$\begin{aligned}
&\left| \frac{\sum_{k=1}^{\infty}(1-\zeta)(q[k]_q+1)\Gamma_{q,k}(\lambda)a_k z^{k+1}}{-2(1-\alpha) + \sum_{k=1}^{\infty}\{q[k]_q[1+(1-2\alpha)\zeta] + (1-2\alpha)(\zeta-1)\}\Gamma_{q,k}(\lambda)a_k z^{k+1}} \right| \\
\leq\ & \frac{\sum_{k=1}^{\infty}(1-\zeta)(q[k]_q+1)\Gamma_{q,k}(\lambda)a_k r^{k+1}}{2(1-\alpha) - \sum_{k=1}^{\infty}\{q[k]_q[1+(1-2\alpha)\zeta] + (1-2\alpha)(\zeta-1)\}\Gamma_{q,k}(\lambda)a_k r^{k+1}} \\
\leq\ & 1.
\end{aligned} \tag{19}$$

Since (19) holds for all r, $0 < r < 1$ letting $r \to 1^-$, we have $\mathcal{F} \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$.

Now, let $\mathcal{F} \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$, since $Re(z) \leq |z|$ for all z. Then

$$\begin{aligned}
&Re\left\{ \frac{zq\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z))}{(\zeta-1)\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z) + q\zeta z\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z))} \right\} \\
&= Re\left\{ \frac{-1 + \sum_{k=1}^{\infty} q[k]_q\Gamma_{q,k}(\lambda)a_k z^{k+1}}{-1 + \sum_{k=1}^{\infty}[\zeta(1+q[k]_q) - 1]\,\Gamma_{q,k}(\lambda)a_k z^{k+1}} \right\} > \alpha.
\end{aligned}$$

Choose values of z on real axis so that $\frac{zq\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z))}{(\zeta-1)\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z)+q\zeta z\partial_q(\mathcal{M}^{\lambda}_{l,s,q}\mathcal{F}(z))}$ is real. Letting $z \to 1$ through positive values, we have (18).

Corollary 2. *If* $\mathcal{F} \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$, *then we have*

$$a_k \leq \frac{1-\alpha}{[q[k]_q(1-\alpha\zeta)+\alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)}. \tag{20}$$

The result is sharp for the function $\mathcal{F}_k(z)$ *defined by*

$$\mathcal{F}_k(z) = \frac{1}{z} + \frac{1-\alpha}{[q[k]_q(1-\alpha\zeta)+\alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)} z^k, \tag{21}$$

for $k \geq 1$.

Theorem 3. *If* $\mathcal{F} \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$, *then*

$$\sum_{k=1}^{\infty} a_k \leq \frac{1-\alpha}{[q(1-\alpha\zeta)+\alpha(1-\zeta)]\,\Gamma_{q,1}(\lambda)}. \tag{22}$$

Proof. Let $\mathcal{F} \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$. Then, in view of (18), we have

$$[q(1-\alpha\zeta)+\alpha(1-\zeta)]\,\Gamma_{q,1}(\lambda) \sum_{k=1}^{\infty} a_k \leq (1-\alpha),$$

we have the assertion (22).

Theorem 4. *Let the function* $\mathcal{F}(z) \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$. *Then*

$$\frac{1}{|z|} - \frac{1-\alpha}{[q(1-\alpha\zeta)+\alpha(1-\zeta)]\,\Gamma_{q,1}(\lambda)} |z| \leq |\mathcal{F}(z)| \leq \frac{1}{|z|} + \frac{1-\alpha}{[q(1-\alpha\zeta)+\alpha(1-\zeta)]\,\Gamma_{q,1}(\lambda)} |z|. \tag{23}$$

The result is sharp.

Proof. For $\mathcal{F}(z) \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$. Then

$$|\mathcal{F}(z)| = \left| \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \right| \leq \frac{1}{|z|} + |z| \sum_{k=1}^{\infty} a_k,$$

and

$$|\mathcal{F}(z)| = \left| \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \right| \geq \frac{1}{|z|} - |z| \sum_{k=1}^{\infty} a_k,$$

which in view of (22), we have (23).

Theorem 5. *Let $\mathcal{F}(z) \in \Phi^{\lambda}_{l,s,q}(\zeta, \alpha)$. Then $\mathcal{F}(z)$ is starlike in $0 < |z| < r_1$, where r_1 is the largest value for which*

$$\frac{([k]_q + 2)(1-\alpha)}{[q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)} r_1^{k+1} \leq 1, \tag{24}$$

for $k \geq 1$. The result is sharp for the function $\mathcal{F}_k(z)$ given by (21).

Proof. It is sufficent to show that

$$\left|\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + 1\right| < 1, \tag{25}$$

we have

$$\left|\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + 1\right| \leq \frac{\sum_{k=1}^{\infty}([k]_q + 1)a_k\, |z|^k}{\frac{1}{|z|} - \sum_{k=1}^{\infty} a_k\, |z|^k}. \tag{26}$$

Hence for $0 < |z| < r$, (26) hold true if

$$\sum_{k=1}^{\infty}([k]_q + 2)a_k r^{k+1} < 1,$$

and by (18), we may take

$$\sum_{k=1}^{\infty} a_k \leq \frac{1-\alpha}{[q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)}\lambda_k, \quad (k \geq 1),$$

where $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k \leq 1$.

For each fixed r, we choose the positive integer $k_0 = k_0(r)$ for which

$$\frac{([k_0]_q + 2)}{[q[k_0]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k_0}(\lambda)} r^{k_0+1}, \quad \text{is maximal.}$$

Then it follows that

$$\sum_{k=1}^{\infty}([k]_q + 2)a_k r^{k+1} \leq \frac{([k]_q + 2)(1-\alpha)}{[q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)} r^{k+1},$$

then $\mathcal{F}$ is starlike in $0 < |z| < r_1$ provided that

$$\frac{([k_0]_q + 2)(1-\alpha)}{[q[k_0]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k_0}(\lambda)} r_1^{k_0+1} \leq 1.$$

We find the value $r_1 = r_0$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{([k_0]_q + 2)(1-\alpha)}{[q[k_0]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k_0}(\lambda)} r_0^{k_0+1} = 1. \tag{27}$$

Then this value is the radius of starlikeness for function $\mathcal{F}$ belong to class $\Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$.

Theorem 6. *Let $\mathcal{F}(z) \in \Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$. Then $\mathcal{F}(z)$ is convex in $0 < |z| < r_2$, where r_2 is the largest value for which*

$$\frac{[k]_q([k-1]_q + 3)(1-\alpha)}{[q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)} r_2^{k+1} \leq 1, \tag{28}$$

for $k \geq 1$. The result is sharp for the function $\mathcal{F}(z)$ given by (21).

Proof. By using the same technique in the proof of Theorem 4 we can show that

$$\left| \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} + 2 \right| < 1, \tag{29}$$

for $0 < |z| < r_2$ with the aid of Theorem 1. Thus, we have the assertion of Theorem 6.

Let the function $\mathcal{F}_j(z)$ be given by

$$\mathcal{F}_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k, \; j = 1, 2, ..., m. \tag{30}$$

Theorem 7. *Let the function $\mathcal{F}_j(z)$ defined by (30) be in the class $\Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$, for each $j = 1, 2, ..., m$, then the function $\mathbb{F}(z)$ defined by*

$$\mathbb{F}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \tag{31}$$

also be in the class $\Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$, where

$$b_k = \frac{1}{m} \sum_{j=1}^{m} a_{k,j}. \tag{32}$$

Proof. Since $\mathcal{F}_j(z) \in \Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$, it follows from Theorem 1, that

$$\sum_{k=1}^{\infty} \left[q[k]_q(1-\alpha\zeta)+\alpha(1-\zeta)\right] \Gamma_{q,k}(\lambda) a_{k,j} \leq 1-\alpha, \; j=1,2,...,m. \tag{33}$$

Hence

$$\begin{aligned} & \sum_{k=1}^{\infty} \left[q[k]_q(1-\alpha\zeta)+\alpha(1-\zeta)\right] \Gamma_{q,k}(\lambda) b_k \\ = & \sum_{k=1}^{\infty} \left[q[k]_q(1-\alpha\zeta)+\alpha(1-\zeta)\right] \Gamma_{q,k}(\lambda) (\frac{1}{m}\sum_{j=1}^{m} a_{k,j}) \\ = & \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=1}^{\infty} \left[q[k]_q(1-\alpha\zeta)+\alpha(1-\zeta)\right] \Gamma_{q,k}(\lambda) a_{k,j}\right) \leq 1-\alpha. \end{aligned}$$

By Theorem 1, we have $\mathbb{F}(z) \in \Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$.

Theorem 8. *The class $\Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$ is closed under convex linear compination.*

Proof. Let $\mathcal{F}_j(z)$ be defined by (30). Define the function $h(z)$ by

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \quad b_k \geq 1. \tag{34}$$

Suppose that $\mathcal{F}(z)$ and $h(z)$ are in the class $\Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$, we only need to prove that

$$G(z) = \xi\mathcal{F}(z) + (1-\xi)h(z) \quad (0 \leq \xi \leq 1), \tag{35}$$

also be in the class. Since

$$G(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left\{\xi a_k + (1-\xi) b_k\right\} z^k, \tag{36}$$

then

$$\sum_{k=1}^{\infty} \left[q[k]_q(1-\alpha\zeta)+\alpha(1-\zeta)\right] \Gamma_{q,k}(\lambda) \left\{\xi a_k + (1-\xi) b_k\right\} \leq (1-\alpha), \tag{37}$$

with the aid of Theorem 1. Hence $G(z) \in \Phi^{\lambda}_{l,s,q}(\zeta,\alpha)$. This clearly completes the proof of the Theorem.

Theorem 9. *Let $\mathcal{F}_0(z) = \frac{1}{z}$ and $\mathcal{F}_k(z)$ defined by (21) for $k \geq 1$. Then the function $\mathcal{F}(z) \in \Phi_{l,s,q}^{\lambda}(\zeta, \alpha)$ if and only if it can be expressed in the form*

$$\mathcal{F}(z) = \sum_{k=0}^{\infty} \eta_k \mathcal{F}_k(z), \tag{38}$$

where $\eta_k \geq 0$ and

$$\sum_{k=0}^{\infty} \eta_k \leq 1. \tag{39}$$

Proof. We suppose that the function $\mathcal{F}(z)$ can be expressed in the form (38). Then from (21) and (39) we have

$$\mathcal{F}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\alpha)\eta_k}{[q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)} z^k, \tag{40}$$

Since

$$\begin{aligned} & \sum_{k=1}^{\infty} [q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda) . \frac{(1-\alpha)\eta_k}{[q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)} \\ = \ & (1-\alpha) \sum_{k=1}^{\infty} \eta_k \\ \leq \ & (1-\alpha). \end{aligned} \tag{41}$$

It follows from Theorem 2 that the function $\mathcal{F}(z) \in \Phi_{l,s,q}^{\lambda}(\zeta, \alpha)$.

Conversely, let $\mathcal{F}(z) \in \Phi_{l,s,q}^{\lambda}(\zeta, \alpha)$ which satisfies (22) for $k \geq 1$, we obtain

$$\eta_k = \frac{(1-\alpha)}{[q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda)} a_k \leq 1,$$

and

$$\eta_0 = 1 - \sum_{k=1}^{\infty} \eta_k.$$

This completes the proof of the Theorem 9.

Corollary 10. *The extreme points of the class $\Phi_{l,s,q}^{\lambda}(\zeta, \alpha)$ are the functions $\mathcal{F}_k(z)$ $(k \geq 1)$ given by (21) in Theorem 9.*

For $\mathcal{F}(z) \in \Phi_\delta$, given by 2, the sequence of partial sums is given by

$$\mathcal{F}_n(z) = \frac{1}{z} + \sum_{k=1}^{n} a_k z^k \ (n \in \mathbb{N}). \tag{42}$$

Now we will follow the work of [16], [11] and [4] on partial sums of meromorphic univalent functions, to obtain the results. Let

$$\Psi_{q,k}^{\lambda}(\alpha,\zeta) = [q[k]_q(1-\alpha\zeta) + \alpha(1-\zeta)]\,\Gamma_{q,k}(\lambda). \tag{43}$$

Theorem 11. *If $\mathcal{F}(z) \in \Phi_\delta$, satisfies the condition (18), then*

$$Re\left(\frac{\mathcal{F}(z)}{\mathcal{F}_n(z)}\right) \geq \frac{\Psi_{q,n+1}^{\lambda} - 1 + \alpha}{\Psi_{q,n+1}^{\lambda}}, \tag{44}$$

where

$$\Psi_{q,k}^{\lambda}(\alpha,\zeta) \geq \begin{cases} 1-\alpha, & if\ k = 1,2,3,...,n \\ \Psi_{q,n+1}^{\lambda}, & if\ k = n+1, n+2, ... \end{cases}. \tag{45}$$

The result (44) is sharp for

$$\mathcal{F}(z) = \frac{1}{z} + \frac{1-\alpha}{\Psi_{q,n+1}^{\lambda}} z^{n+1}. \tag{46}$$

Proof. Let

$$\begin{aligned} \frac{1+\omega(z)}{1-\omega(z)} &= \frac{\Psi_{q,n+1}^{\lambda}}{1-\alpha}\left[\frac{\mathcal{F}(z)}{\mathcal{F}_n(z)} - \frac{\Psi_{q,n+1}^{\lambda} - 1 + \alpha}{\Psi_{q,n+1}^{\lambda}}\right] \\ &= \frac{1 + \sum_{k=1}^{n} a_k z^{k+1} + \left(\frac{\Psi_{q,n+1}^{\lambda}}{1-\alpha}\right)\sum_{k=n+1}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{n} a_k z^{k+1}}. \end{aligned} \tag{47}$$

It suffices to show that $|\omega(z)| \leq 1$. Now from (47) we have

$$\omega(z) = \frac{\left(\frac{\Psi_{q,n+1}^{\lambda}}{1-\alpha}\right)\sum_{k=n+1}^{\infty} a_k z^{k+1}}{2 + 2\sum_{k=1}^{n} a_k z^{k+1} + \left(\frac{\Psi_{q,n+1}^{\lambda}}{1-\alpha}\right)\sum_{k=n+1}^{\infty} a_k z^{k+1}}.$$

Hence we obtain

$$|\omega(z)| \leq \frac{\left(\frac{\Psi_{q,n+1}^{\lambda}}{1-\alpha}\right)\sum_{k=n+1}^{\infty} a_k}{2 - 2\sum_{k=1}^{n} a_k - \left(\frac{\Psi_{q,n+1}^{\lambda}}{1-\alpha}\right)\sum_{k=n+1}^{\infty} a_k}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$2\left(\frac{\Psi^{\lambda}_{q,n+1}}{1-\alpha}\right)\sum_{k=n+1}^{\infty} a_k \leq 2-2\sum_{k=1}^{n} a_k,$$

or, equivalently

$$\sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty}\left(\frac{\Psi^{\lambda}_{q,n+1}}{1-\alpha}\right) a_k \leq 1.$$

From (18), it is sufficient to show that

$$\sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty}\left(\frac{\Psi^{\lambda}_{q,n+1}}{1-\alpha}\right) a_k \leq \sum_{k=1}^{\infty}\left(\frac{\Psi^{\lambda}_{q,k}}{1-\alpha}\right) a_k,$$

which is equivalent to

$$\sum_{k=1}^{n}\left(\frac{\Psi^{\lambda}_{q,k}-1+\alpha}{1-\alpha}\right) a_k + \sum_{k=n+1}^{\infty}\left(\frac{\Psi^{\lambda}_{q,k}-\Psi^{\lambda}_{q,n+1}}{1-\alpha}\right) a_k \geq 0. \tag{48}$$

For $z = re^{i\pi/n}$ we have

$$\frac{\mathcal{F}(z)}{\mathcal{F}_n(z)} = 1 + \frac{1-\alpha}{\Psi^{\lambda}_{q,n+1}} z^k \to 1 - \frac{1-\alpha}{\Psi^{\lambda}_{q,n+1}} z^k = \frac{\Psi^{\lambda}_{q,n+1}-1+\alpha}{\Psi^{\lambda}_{q,n+1}} \text{ where } r \to 1^{-},$$

which shows that $\mathcal{F}(z)$ given by (46) gives the sharpness.

Theorem 12. *If $\mathcal{F}(z) \in \Phi_\delta$, satisfies the condition (18), then*

$$Re\left(\frac{\mathcal{F}_n(z)}{\mathcal{F}(z)}\right) \geq \frac{\Psi^{\lambda}_{q,n+1}}{\Psi^{\lambda}_{q,n+1}+1-\alpha}, \tag{49}$$

where $\Psi^{\lambda}_{q,n+1}$ is defined by (43) and satisfies (45) and $\mathcal{F}(z)$ given by (46) gives the sharpness.

Proof. The proof follows by defining

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{\Psi^{\lambda}_{q,n+1}+1-\alpha}{1-\alpha}\left[\frac{\mathcal{F}_n(z)}{\mathcal{F}(z)} - \frac{\Psi^{\lambda}_{q,n+1}}{\Psi^{\lambda}_{q,n+1}+1-\alpha}\right]. \tag{50}$$

The reminder part is as in Theorem 11. So, we omit it.

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