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A NOTE ON HELICES OF MINKOWSKI SPACE

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Abstract. The curve α is called a general helix if $\langle V_1, W\rangle$ is a constant function, where W is a constant vector field different from zero. We define the second kind of harmonic curvatures and Darboux vector of a non-null unit speed curve and give different characterizations of general helices with this curvatures and with the second kind of Darboux vector.

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1. Introduction

Helix is a space curve with a lot of work on it. Helices have been focus for a number of authors [1-16]. In 1845, Venant obtained that κ/τ is a constant function iff a curve is a helix [17]. Helices by the fact that the function

$$\left(\frac{\kappa_1}{\kappa_2}\right)^2 + \left(\frac{1}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)'\right)^2$$

is constant with the second curvature κ_2 and the third curvature κ_3 into E^4. See also [12].

In this work we study general helices with the second kind of harmonic curvatures in Minkowski space. We consider Minkowski space E^n_ν with Lorentzian metric

$$\langle\ ,\rangle = -\sum_{i=1}^{\nu} dx_i^2 + \sum_{i=\nu+1}^{n} dx_i^2,$$

where $(x_1, ..., x_n)$ is a coordinate system of $\mathbb{R}^n$. Let be $w \in E^n_\nu$.

1. If$\langle w, w\rangle > 0$ or $w = 0$, then the vector w is called spacelike.

2. If $\langle w, w\rangle < 0$, then the vector w is called timelike.

3. If $\langle w, w\rangle = 0$ and $w \neq 0$, then the vector w is called lightlike.
The arbitrary timelike vectors u and w are in the same timecone iff $\langle u, w\rangle < 0$. The magnitude of a vector w is defined by $\|w\| = \sqrt{|\langle w, w\rangle|}$ [18].

Let $\alpha : I \subset \mathbb{R} \to E_\nu^n$ be a regular curve, i.e. $\alpha'(s) \neq 0$, where $\alpha'(s) = d\alpha/dt$. The curve α is named as

1. spacelike, if $\alpha'(s)$ is spacelike for all $s \in I$.
2. timelike, if $\alpha'(s)$ is timelike for all $s \in I$.
3. null(lightlike), if $\alpha'(s)$ is null(lightlike) for all $s \in I$.

If α is spacelike or timelike, then α is called a non-null curve. We parametrize a non-null or null curve α with the pseudo-arc length parameter t, if $\langle \alpha'(t), \alpha'(t)\rangle = \pm 1$ or $\langle \alpha''(t), \alpha''(t)\rangle = 1$, respectively. In either case α is a unit speed curve [18]. In whole this article we use non-null curve with the pseudo-arc length parameter. For the sake of simplicity, in the whole article we will understand the non-null curve with the pseudo-arc length parameter curves, when we say curve.

Definition 1.1. We assume that $\alpha : I \to E_\nu^n, I \subset \mathbb{R}$ is a curve and $\{V_1(s), ..., V_n(s)\}$ is the Frenet frame of α. $i{-}th$ curvature of α is $k_i : I \to \mathbb{R}$, $\kappa_i(s) = \varepsilon_{i+1}\left\langle V_i^{'}(s), V_{i+1}(s)\right\rangle$ with $1 \leq i \leq n-1$ and $\varepsilon_i = \langle V_i, V_i\rangle$ [5].

Theorem 1.1. Let α be a curve in E_ν^n with the Frenet frame $\{V_1(s), ..., V_n(s)\}$ and curvature functions k_i. One get the Frenet equations following by

$$\begin{aligned} V_1' &= \kappa_1 V_2 \\ V_i' &= -\varepsilon_{i-1}\varepsilon_i\kappa_{i-1}V_{i-1} + \kappa_i V_{i+1}, \\ V_n' &= -\varepsilon_{n-1}\varepsilon_n\kappa_{n-1}V_{n-1} \end{aligned} \tag{1}$$

where $\varepsilon_i = \langle V_i, V_i\rangle = \pm 1$ and $1 \leq i \leq n-1$ [5].

For special case we assume that $\alpha = \alpha(s)$ is a curve in E_1^3, $\{T, N, B\}$ the Frenet frame and κ_i be $i - th$ curvature functions of the curve $(i = 1, 2)$. Then the Frenet equations are given as

$$\begin{aligned} T' &= \kappa_1 V_2 \\ N' &= -\varepsilon_1\varepsilon_2\kappa_1 T + \kappa_2 B \\ B' &= -\varepsilon_2\varepsilon_3\kappa_2 N \end{aligned}$$

with $\langle T, T\rangle = \varepsilon_1 = \pm 1$, $\langle N, N\rangle = \varepsilon_2 = \pm 1$ and $\langle B, B\rangle = \varepsilon_3 = \pm 1$. Moreover the curvature functions of the curve α is following

$$\kappa_1 = \varepsilon_2\left\langle T^{'}, N\right\rangle, \qquad \kappa_2 = \varepsilon_3\left\langle N^{'}, B\right\rangle.$$

Definition 1.2. The angles between the vectors y and z in the Minkowski space are defined following:

1. If y and z are timelike vectors such that they lying in the same timecone, then $< y, z >= -\|y\| \|z\| \cosh \phi$ with a unique real number $\phi \geq 0$. ϕ is named as the hyperbolic angle.

2. If y and z are spacelike vectors such that they span a timelike vector space, then $|< y, z >| = \|y\| \|z\| \cosh \phi$ with a unique real number $\phi \geq 0$. ϕ is named as the central angle.

3. If y and z are spacelike vectors such that they span spacelike vector space, then $< y, z >= \|y\| \|z\| \cos \phi$ with a unique real number $0 < \phi < \pi$. ϕ is named as the spacelike angle.

4. If y is a spacelike vector and z is a timelike vector, then $|< y, z >| = \|y\| \|z\| \sinh \phi$ with a unique real number $\phi \geq 0$. ϕ is named as the Lorentzian timelike angle [4, 18].

2. Second kind of harmonic curvatures with general helices in E_v^n

Ekmekçi et. al. in [6] gave harmonic curvatures of a curve following:

Definition 2.1. Harmonic curvatures $H_i : I \to \mathbb{R}$, $I \subset \mathbb{R}$, $1 \leq i \leq n-1$ of a curve $\alpha : I \to E_\nu^n$ are defined following

$$H_i = \begin{cases} 0, & i = 0, \\ \varepsilon_1 \varepsilon_2 \dfrac{\kappa_1}{\kappa_2}, & i = 1, \\ \dfrac{1}{\kappa_{i+1}} \left[\varepsilon_i \varepsilon_{i+1} \kappa_i H_{i-2} + H'_{i-1} \right], & i = 2, 3, ..., n-2. \end{cases} \tag{2}$$

with non-zero curvatures κ_i, $1 \leq i \leq n-1$.

We refer to functions H_i as *the first kind of harmonic curvatures* of the curve. Now, we obtain several characterizations for general helix by using the new functions S_i called *the second kind of harmonic curvatures* of the curve.

Definition 2.2. If the function $< V_1, W >$ is constant for tangent vector field V_1 of a curve $\alpha : I \to E_\nu^n$ and a different from zero constant vector field W, then the curve α is called general helix.

Theorem 2.1. A curve α is a general helix in E_ν^n iff there exist differentiable functions $S_i : I \to \mathbb{R}$, $I \subset \mathbb{R}$, $1 \le i \le n$ satisfying the equations

$$S_i = \begin{cases} 1, & i = 1, \\ 0, & i = 2, \\ \dfrac{\varepsilon_{i-1}\varepsilon_i}{\kappa_{i-1}} \left[\kappa_{i-2} S_{i-2} + S'_{i-1}\right], & i = 3, 4, ..., n. \end{cases} \tag{3}$$

with the condition

$$S'_n = -\kappa_{n-1} S_{n-1}. \tag{4}$$

Proof. We assume that α is a general helix. Then the function $\langle V_1, W \rangle$ is a constant for a fixed axis W. Consider the differentiable vector field

$$W = \sum_{i=1}^{n} w_i V_i, \tag{5}$$

where

$$w_i = \varepsilon_i \langle V_i, W \rangle, \qquad 1 \le i \le n \tag{6}$$

are differentiable functions. Since α is a general helix, then the function $w_1 = \varepsilon_1 \langle V_1, W \rangle$ is constant. If we differentiate (6) with respect to s and from the equations (1), then one obtain

$$w'_1(s) = \varepsilon_1 \varepsilon_2 \kappa_1 w_2 = 0.$$

If $w_2 = 0$ and the vector field W is constant, then $W \in sp\{V_1, V_3, ..., V_n\}$. Since the vector field W is constant, by differentiating the equation (5) and using (1), then we obtain the O.D.E system

$$\left.\begin{array}{c} -\kappa_1 w_1 + \varepsilon_2 \varepsilon_3 \kappa_2 w_3 = 0 \\ w'_3 - \varepsilon_3 \varepsilon_4 \kappa_3 w_4 = 0 \\ w'_4 + \kappa_3 w_3 - \varepsilon_4 \varepsilon_5 \kappa_4 w_5 = 0 \\ \vdots \\ w'_{n-1} + \kappa_{n-2} w_{n-2} - \varepsilon_{n-1} \varepsilon_n \kappa_{n-1} w_n = 0 \\ w'_n + \kappa_{n-1} w_{n-1} = 0. \end{array}\right\} \tag{7}$$

Let be

$$w_j = S_j w_1, \qquad 3 \le j \le n. \tag{8}$$

The functions $S_j : I \to \mathbb{R}$ for $3 \le j \le n$ are differentiable. It must be $w_1 \ne 0$, otherwise from (7) it would be $w_j = 0$, for $3 \le j \le n$. Hence $W = 0$ and this is a contradiction. According to (7), we obtain

$$\left.\begin{array}{lll} S_3 & = & \varepsilon_2\varepsilon_3\frac{\kappa_1}{\kappa_2} \\ S_4 & = & \frac{\varepsilon_3\varepsilon_4}{\kappa_3}S_3' \\ S_5 & = & \frac{\varepsilon_4\varepsilon_5}{\kappa_4}[\kappa_3 S_3 + S_4'] \\ \vdots & & \\ S_{n-1} & = & \frac{\varepsilon_{n-2}\varepsilon_{n-1}}{\kappa_{n-2}}[\kappa_{n-3}S_{n-3} + S_{n-2}'] \\ S_n & = & \frac{\varepsilon_{n-1}\varepsilon_n}{\kappa_{n-1}}[\kappa_{n-2}S_{n-2} + S_{n-1}'] \end{array}\right\} \tag{9}$$

At the end of (7) one obtain (4). Conversely, we assume that α is a curve with differentiable functions S_j for $1 \leq j \leq n$ satisfying the equations (3) and (4). Consider the unit vector field W is defined by the following equation

$$W = w_1\left[V_1 + \sum_{j=3}^{n} S_j V_j\right]$$

with $w_1 \in \mathbb{R}$. If we differentiate W and use the equations (3) and (4), then we obtain $W' = 0$. Also W is a constant vector field and $< V_1, W >= \varepsilon_1 w_1$ is a constant function. Therefore the curve α is a general helix.

Now, we are in a position to define the second kind of harmonic curvatures of a curve.

Definition 2.3. Let $\alpha : I \to E_\nu^n$ be a curve with non-zero curvatures k_i $(i = 1, 2, ..., n-1)$. We define the *second kind of harmonic curvatures* of α denoted by $S_i : I \subset \mathbb{R} \to \mathbb{R}$, $i = 1, 2, ..., n$, given by the equation (3) such that

$$S_i = \begin{cases} 1, & i = 1, \\ 0, & i = 2, \\ \frac{\varepsilon_{i-1}\varepsilon_i}{\kappa_{i-1}}\left[\kappa_{i-2}S_{i-2} + S_{i-1}'\right], & i = 3, 4, ..., n. \end{cases}$$

Corollary 2.1. A curve α is a general helix in E_ν^n iff the second kind of harmonic curvatures S_n and S_{n-1} satisfy the equation (4), that is

$$S_n' = -\kappa_{n-1}S_{n-1}.$$

By making the variation of parameter, we get different characterization

$$u(t) = \int_0^t \kappa_{n-1}(x)dx, \qquad \frac{du}{dt} = \kappa_{n-1}(t).$$

Since $S_n' = -\kappa_{n-1}S_{n-1}$ in the equation (4), one obtain the following equation

$$S_{n-1}'(u) = \varepsilon_{n-1}\varepsilon_n S_n(u) - \left(\frac{\kappa_{n-2}(u)}{\kappa_{n-1}(u)}\right) S_{n-2}(u).$$

Substitutite this equation into (3), we get the equation

$$S_n''(u) + \varepsilon_{n-1}\varepsilon_n S_n(u) = \frac{\kappa_{n-2}(u)S_{n-2}(u)}{\kappa_{n-1}(u)}.$$

Making change of variables, depending on the value of $\varepsilon_{n-1}\varepsilon_n$, we have two general solution of this equation:

1) If $\varepsilon_{n-1}\varepsilon_n = 1$, then

$$S_n(u) = \left(m - \int \frac{\kappa_{n-2}(u)S_{n-2}(u)}{\kappa_{n-1}(u)} \sin u du\right) \cos u + \left(n + \int \frac{\kappa_{n-2}(u)S_{n-2}(u)}{\kappa_{n-1}(u)} \cos u du\right) \sin u$$

where m and n are arbitrary constants. Also this solution is the same for any general helix in Euclidean space [1]. Because of that in this paper we give proofs for only the following solution.

2) If $\varepsilon_{n-1}\varepsilon_n = -1$, then

$$S_n(u) = \left(m - \int \frac{\kappa_{n-2}(u)S_{n-2}(u)}{\kappa_{n-1}(u)} \sinh u du\right) \cosh u + \left(n + \int \frac{\kappa_{n-2}(u)S_{n-2}(u)}{\kappa_{n-1}(u)} \cosh u du\right) \sinh u \tag{10}$$

where m and n are arbitrary constants. From the equation (10), we get

$$\begin{aligned} S_n(t) \quad = \quad & \left(m - \int \left[\kappa_{n-2}(t)S_{n-2}(t) \sinh \int \kappa_{n-1}(t)dt\right] dt\right) \cosh \int^t \kappa_{n-1}(x)dx \\ & + \left(n + \int \left[\kappa_{n-2}(t)S_{n-2}(t) \cosh \int \kappa_{n-1}(t)dt\right] dt\right) \sinh \int^t \kappa_{n-1}(x)dx. \end{aligned} \tag{11}$$

According to (4), we obtain

$$\begin{aligned} S_{n-1}(t) \quad = \quad & \frac{-S_n'(t)}{\kappa_{n-1}(t)} \\ = \quad & \left(-m + \int \left[\kappa_{n-2}(t)S_{n-2}(t) \sinh \int \kappa_{n-1}(t)dt\right] dt\right) \sinh \int^t \kappa_{n-1}(x)dx \\ & - \left(n + \int \left[\kappa_{n-2}(t)S_{n-2}(t) \cosh \int \kappa_{n-1}(t)dt\right] dt\right) \cosh \int^t \kappa_{n-1}(x)dx. \end{aligned} \tag{12}$$

From Corollary 2.1, we can give the following theorems.

Theorem 2.2. We assume that $\alpha : I \to E_\nu^n$ is parameterized the pseudo-arc length parameter t with $\varepsilon_{n-1}\varepsilon_n = -1$. Then α is a general helix iff

$$\begin{aligned} S_{n-1}(t) \quad = \quad & \left(-m + \int \left[\kappa_{n-2}(t)S_{n-2}(t) \sinh \int \kappa_{n-1}(t)dt\right] dt\right) \sinh \int^t \kappa_{n-1}(x)dx \\ & - \left(n + \int \left[\kappa_{n-2}(t)S_{n-2}(t) \cosh \int \kappa_{n-1}(t)dt\right] dt\right) \cosh \int^t \kappa_{n-1}(x)dx \end{aligned} \tag{13}$$

where m and n are constants.

Proof. Suppose that α is a general helix. Let us define $f(t)$ and $g(t)$ by

$$\begin{array}{rcl} f(t) & = & S_n(t)\cosh\phi + S_{n-1}(t)\sinh\phi + \int \kappa_{n-2}(t)S_{n-2}(t)\sinh\phi dt \\ g(t) & = & -S_n(t)\sinh\phi - S_{n-1}(t)\cosh\phi - \int \kappa_{n-2}(t)S_{n-2}(t)\cosh\phi dt \end{array} \quad (14)$$

where

$$\phi(t) = \int^t \kappa_{n-1}(x)dx,$$

and the functions S_{n-2}, S_{n-1}, S_n are as in Theorem 2.1. If we differentiate equations (14) with respect to t and taking into account of (3) and (4), we obtain

$$\begin{array}{rcl} \dfrac{df}{du} & = & S_n'\cosh\phi + S_n\kappa_{n-1}\sinh\phi + S_{n-1}'\sinh\phi \\ & & +S_{n-1}\kappa_{n-1}\cosh\phi + \kappa_{n-2}S_{n-2}\sinh\phi \\ & = & -\kappa_{n-1}S_{n-1}\cosh\phi + S_n\kappa_{n-1}\sinh\phi - (\kappa_{n-1}S_n + \kappa_{n-2}S_{n-2})\sinh\phi \\ & & +S_{n-1}\kappa_{n-1}\cosh\phi + \kappa_{n-2}S_{n-2}\sinh\phi \\ & = & 0 \end{array}$$

and

$$\begin{array}{rcl} \dfrac{dg}{dt} & = & -S_n'\sinh\phi - S_n\kappa_{n-1}\cosh\phi - S_{n-1}'\cosh\phi \\ & & -S_{n-1}\kappa_{n-1}\sinh\phi - \kappa_{n-2}S_{n-2}\cosh\phi \\ & = & \kappa_{n-1}S_{n-1}\sinh\phi - S_n\kappa_{n-1}\cosh\phi + (\kappa_{n-1}S_n + \kappa_{n-2}S_{n-2})\cosh\phi \\ & & -S_{n-1}\kappa_{n-1}\sinh\phi - \kappa_{n-2}S_{n-2}\cosh\phi \\ & = & 0. \end{array}$$

Also we get $f(t) = m$ and $g(t) = n$ with constants m, n.

$$S_{n-1}(t) = \left(-m + \int [\kappa_{n-2}S_{n-2}\sinh\phi dt]\right)\sinh\phi - \left(n + \int [\kappa_{n-2}S_{n-2}\cosh\phi dt]\right)\cosh\phi.$$

Conversely, we assume that the equation (13) is true. According to Theorem 2.1, $S_n(t)$ is defined by following equation

$$S_n(t) = \left(m - \int [\kappa_{n-2}S_{n-2}\sinh\phi dt]\right)\cosh\phi + \left(n + \int [\kappa_{n-2}S_{n-2}\cosh\phi dt]\right)\sinh\phi$$

with $\phi(t) = \int^t \kappa_{n-1}(x)dx$. A direct differentiation of (13) gives

$$\begin{array}{rcl} S_{n-1}' & = & \kappa_{n-2}S_{n-2}\sinh^2\phi + \left(-m + \int [\kappa_{n-2}S_{n-2}\sinh\phi dt]\right)\kappa_{n-1}\cosh\phi \\ & & -\kappa_{n-2}S_{n-2}\cosh^2\phi - \left(n + \int [\kappa_{n-2}S_{n-2}\cosh\phi dt]\right)\kappa_{n-1}\sinh\phi \\ & = & -\kappa_{n-2}S_{n-2} - \kappa_{n-1}S_n. \end{array}$$

This verifies the equation (3) for $i = n$. In addition we get $S'_n = -\kappa_{n-1} S_{n-1}$, which finishes the proof.

Theorem 2.3. We assume that $\alpha : I \to E^n_\nu$ is parameterized the pseudo-arc length parameter t with $\varepsilon_{n-1}\varepsilon_n = 1$. Then α is a general helix iff

$$\begin{aligned} S_{n-1}(t) \quad = \quad & \left(m - \int \left[\kappa_{n-2} S_{n-2} \sin \int \kappa_{n-1} dt\right] dt\right) \sin \int^t \kappa_{n-1}(x) dx \\ & - \left(n + \int \left[\kappa_{n-2} S_{n-2} \cos \int \kappa_{n-1} dt\right] dt\right) \cos \int^t \kappa_{n-1}(x) dx \end{aligned} \tag{15}$$

where m and n are constants.

3. Second kind of Darboux vector with general helices in E^n_ν

Now we characterize general helices with the second kind of harmonic curvatures and Darboux vector. Firstly, we get an important theorem following:

Theorem 3.1. We suppose that $\alpha : I \to E^n_\nu$ is a curve with its Frenet frame $\{V_1, V_2, ..., V_n\}$ and second kind of harmonic curvatures $\{S_1, S_2, ..., S_n\}$. Then α is a general helix iff

$$\sum_{j=1}^{n} \varepsilon_j S_j^2 = C, \tag{16}$$

where C is a constant different from zero.

Proof. According to Theorem 2.1 and since W is a unit vector, then the proof is obviously.

This theorem gives generalization of $n = 3$ and $n = 4$. Thus, for $n = 3$, from the equation (16), we can write $\frac{\kappa_1}{\kappa_2} = C$. For $n = 4$, from the equation (16), we can write

$$\varepsilon_3 \left(\frac{\kappa_1}{\kappa_2}\right)^2 + \varepsilon_4 \left[\frac{1}{\kappa_3}\left(\frac{\kappa_1}{\kappa_2}\right)'\right]^2 = C.$$

Corollary 3.1. Depending on the angle ϕ, the constant C given by above theorem is

1) If ϕ is a hyperbolic angle, then $C = -\sec h^2 \phi$.
2) If ϕ is a central angle, then $C = \sec h^2 \phi$.
3) If ϕ is a spacelike angle, then $C = \sec^2 \phi$.
4) If ϕ is a Lorentzian angle, then $C = -\varepsilon_1 \csc h^2 \phi$.

Proof. 1) Since V_1 and W are timelike vectors in the same timecone, then

$$w_1 = -\langle V_1, W\rangle = \cosh\phi.$$

From $\langle W, W\rangle = -1$ and (8), with (3), we have

$$C = \sum_{j=1}^{n}\varepsilon_j S_j^2 = -1 + \frac{1}{w_1^2}\sum_{j=3}^{n}\varepsilon_j w_j^2 = -1 + \frac{-1+w_1^2}{w_1^2} = -\sec h^2\phi.$$

Proof of the other cases is similar to above.

All curvatures of a curve are constant different from zero, then the curve is a W-curve [10]. Additionally all the curvature ratios of a curve are constant, then the curve is called a ccr (constant curvature ratios)-curve [9].

Corollary 3.2. Let the curve $\alpha : I \subset \mathbb{R} \to E_\nu^n$ be a general helix. If the curve is a W-curve, then the second kind of harmonic curvatures S_j given in (3) satisfy the following properties:

$$\begin{aligned} S_j &= 0, \quad \text{if } j \text{ is even} \\ S_j &= \prod_{i=1}^{\frac{j-1}{2}}\varepsilon_{2i}\varepsilon_{2i+1}\frac{\kappa_{2i-1}}{\kappa_{2i}}, \quad \text{if } j \text{ is odd } (j \neq 1). \end{aligned} \tag{17}$$

Proof. Let α be a W-curve. From (3) if j is odd

$$\begin{aligned} S_3 &= \varepsilon_2\varepsilon_3\frac{\kappa_1}{\kappa_2} = constant, \\ S_5 &= \frac{\varepsilon_4\varepsilon_5}{\kappa_4}[\kappa_3 S_3 + S_4'] = \varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\frac{\kappa_1\kappa_3}{\kappa_2\kappa_4}, \\ S_7 &= \frac{\varepsilon_6\varepsilon_7}{\kappa_6}[\kappa_5 S_5 + S_6'] = \varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\frac{\kappa_1\kappa_3\kappa_5}{\kappa_2\kappa_4\kappa_6}, \\ &\vdots \\ S_{j-2} &= \frac{\varepsilon_{j-3}\varepsilon_{j-2}}{\kappa_{j-3}}[\kappa_{j-4}S_{j-4} + S_{j-3}'] = \varepsilon_2\varepsilon_3\varepsilon_4...\varepsilon_{j-2}\varepsilon_{j-1}\frac{\kappa_1\kappa_3\kappa_5...\kappa_{j-4}}{\kappa_2\kappa_4\kappa_6...\kappa_{j-3}} \\ S_j &= \frac{\varepsilon_{j-1}\varepsilon_j}{\kappa_{j-1}}[\kappa_{j-2}S_{j-2} + S_{j-1}'] = \varepsilon_2\varepsilon_3\varepsilon_4...\varepsilon_{j-1}\varepsilon_j\frac{\kappa_1\kappa_3\kappa_5...\kappa_{j-2}}{\kappa_2\kappa_4\kappa_6...\kappa_{j-1}} \\ &\vdots \end{aligned}$$

If j is even

$$S_2 = 0, \quad S_4 = \varepsilon_2\varepsilon_3\frac{\varepsilon_3\varepsilon_4}{\kappa_3}(\frac{\kappa_1}{\kappa_2})' = 0, \quad S_6 = 0, ...S_j = 0, ...$$

Therefore from this equations we obtain (17).

Corollary 3.3. Let the curve $\alpha : I \subset \mathbb{R} \to E^n_\nu$ be a general helix. We suppose that the curve is a ccr-curve, then the second kind of harmonic curvatures S_j given in (3) are constant.

Proof. Proof is the same as the proof Corollary 3.2.

Besides, from the definition of the second kind of harmonic curvature functions, we obtain following lemma.

Lemma 3.1. We suppose that $\alpha : I \to E^n_\nu$ is a curve with its Frenet frame $\{V_1, V_2, ..., V_n\}$, and second kind of harmonic curvatures $\{S_1, S_2, ..., S_n\}$. If the curve $\alpha : I \subset \mathbb{R} \to E^n_\nu$ is a general helix, then

$$\varepsilon_j \langle V_j, W \rangle = \varepsilon_1 S_j \langle V_1, W \rangle, \quad 1 \leq j \leq n \tag{18}$$

with W is a fixed axis of the general helix α.

From this lemma, we obtain the following corollary.

Corollary 3.4. If W is an axis of the general helix α, then we can write

$$W = \sum_{j=1}^{n} w_j V_j.$$

From the Lemma 3.4 we get

$$w_j = \varepsilon_j \langle V_j, W \rangle = \varepsilon_1 S_j \langle V_1, W \rangle, 1 \leq j \leq n$$

where $\langle V_1, W \rangle = \varepsilon_1 w_1$ is constant. By the definition of the second kind of harmonic curvatures of the curve, we obtain

$$W = w_1 \left(\sum_{j=1}^{n} S_j V_j \right).$$

Also the vector field

$$D = \sum_{j=1}^{n} S_j V_j$$

is an axis of the general helix α.

Definition 3.1. We suppose that $\alpha : I \to E^n_\nu$ is a curve with its Frenet frame $\{V_1, V_2, ..., V_n\}$ and second kind of harmonic curvatures $\{S_1, S_2, ..., S_n\}$. We call the vector

$$D = \sum_{j=1}^{n} S_j V_j \tag{19}$$

is called the *second kind of Darboux vector* of the curve α.

Theorem 3.2. We assume that $\alpha : I \to E^n_\nu$ is a curve with its Frenet frame $\{V_1, V_2, ..., V_n\}$, and second kind of harmonic curvatures $\{S_1, S_2, ..., S_n\}$. Then α is a general helix iff the second kind of Darboux vector D is a constant.

Proof. If α is a general helix in Minkowski space E^n_ν. Then from Corollary 3.5 we get

$$W = w_1 \left(\sum_{j=1}^{n} S_j V_j \right).$$

Since w_1 is a constant, then D is a constant vector field.

Conversely, let the second kind of Darboux vector D be constant, then we obtain $\langle D, V_1 \rangle = \varepsilon_1$. Also one get $w_1 = \dfrac{1}{\|D\|}$ is constant. For $W = w_1 D$, where $\langle W, V_1 \rangle = \varepsilon_1 w_1$ is constant. Hence W is a constant vector field. So α is a general helix in Minkowski space E^n_ν. This finishes the proof.

From the definitions of S_i in (3), we intend to express the functions S_i in terms of S_3 and the curvatures of α as follows:

$$S_j = \sum_{i=0}^{j-3} A_{ji} S_3^{(i)}, \quad 3 \le j \le n, \tag{20}$$

where

$$S_3^{(i)} = \frac{d^{(i)} S_3}{ds^i}, \qquad S_3^{(0)} = S_3 = \varepsilon_2 \varepsilon_3 \frac{\kappa_1}{\kappa_2}.$$

Then

$$\begin{aligned} S_4 &= \varepsilon_3 \varepsilon_4 \kappa_3^{-1} S_3' = A_{41} S_3' + A_{40} S_3, \\ S_5 &= A_{52} S_3'' + A_{51} S_3' + A_{50} S_3, \\ S_6 &= A_{63} S_3''' + A_{62} S_3'' + A_{61} S_3' + A_{60} S_3 \end{aligned}$$

where

$$\begin{aligned} A_{41} &= \varepsilon_3 \varepsilon_4 \kappa_3^{-1}, \quad A_{40} = 0, \\ A_{52} &= \varepsilon_3 \varepsilon_5 \kappa_4^{-1} \kappa_3^{-1}, \quad A_{51} = \varepsilon_3 \varepsilon_5 \kappa_4^{-1} (\kappa_3^{-1})', \quad A_{50} = \varepsilon_4 \varepsilon_5 \kappa_4^{-1} \kappa_3, \\ A_{63} &= \varepsilon_3 \varepsilon_6 \kappa_5^{-1} \kappa_4^{-1} \kappa_3^{-1}, \quad A_{62} = \varepsilon_3 \varepsilon_6 \kappa_5^{-1} [\kappa_4^{-1} (\kappa_3^{-1})' + (\kappa_4^{-1} \kappa_3^{-1})'], \\ A_{61} &= \varepsilon_5 \varepsilon_6 \kappa_5^{-1} [\varepsilon_3 \varepsilon_4 \kappa_4 \kappa_3^{-1} + \varepsilon_3 \varepsilon_5 (\kappa_4^{-1} (\kappa_3^{-1})')' + \varepsilon_4 \varepsilon_5 \kappa_4^{-1} \kappa_3], \quad A_{60} = \varepsilon_4 \varepsilon_6 \kappa_5^{-1} (\kappa_4^{-1} \kappa_3)' \end{aligned}$$

and so on. Define the following functions:

$$A_{30} = 1, \quad A_{40} = 0$$

$$A_{j0} = \varepsilon_{j-1}\varepsilon_j \left[\kappa_{j-1}^{-1}\kappa_{j-2}A_{(j-2)0} + \kappa_{j-1}^{-1}A'_{(j-1)0}\right], \quad 5 \le j \le n$$

$$A_{j(j-3)} = \varepsilon_3\varepsilon_j\kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}...\kappa_4^{-1}\kappa_3^{-1}, \qquad 4 \le j \le n$$

$$A_{j(j-4)} = \varepsilon_3\varepsilon_j[\kappa_{j-1}^{-1}\left(\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}...\kappa_4^{-1}\kappa_3^{-1}\right)' + \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\left(\kappa_{j-3}^{-1}...\kappa_4^{-1}\kappa_3^{-1}\right)'$$
$$+... + \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}...\kappa_4^{-1}(\kappa_3^{-1})'], \qquad 5 \le j \le n$$

$$A_{ji} = \varepsilon_{j-1}\varepsilon_j \left[\kappa_{j-1}^{-1}\kappa_{j-2}^{-1}A_{(j-2)i} + \kappa_{j-1}^{-1}(A'_{(j-1)i} + A_{(j-1)(i-1)})\right], \qquad 1 \le i \le j-5, \;\; 6 \le j \le n$$

and $A_{ji} = 0$ otherwise.

As a consequence of Theorem 2.2, according to the functions A_{ji}, we have the following equation. (4) leads the followig condition

$$\begin{aligned} A_{n(n-3)}S_3^{(n-2)} \quad & + \left(A'_{n(n-3)} + A_{n(n-4)}\right) S_3^{(n-3)} \\ & + \sum_{i=1}^{n-4} \left[A'_{ni} + A_{n(i-1)} + \kappa_{n-1}A_{(n-1)i}\right] S_3^{(i)} \\ & + (A'_{n0} + \kappa_{n-1}A_{(n-1)0})S_3 = 0, \qquad n \ge 3. \end{aligned} \tag{21}$$

As a consequence of (21) and Theorem 2.1 according to the functions A_{ji}, one can write this corollary.

Corollary 3.5. The properties are equivalent:

1. α is a general helix.
2. For $n \ge 3$

$$\begin{aligned} 0 \;\; = \;\; & A_{n(n-3)}\left(\frac{\kappa_1}{\kappa_2}\right)^{(n-2)} + \left(A'_{n(n-3)} + A'_{n(n-4)}\right)\left(\frac{\kappa_1}{\kappa_2}\right)^{(n-3)} \\ & + \sum_{i=1}^{n-4}\left[A'_{ni} + A_{n(i-1)} + \kappa_{n-1}A_{(n-1)i}\right]\left(\frac{\kappa_1}{\kappa_2}\right)^{(i)} \\ & + \left(A'_{n0} + \kappa_{n-1}A_{(n-1)0}\right)\left(\frac{\kappa_1}{\kappa_2}\right). \end{aligned}$$

3. The function

$$\sum_{j=3}^{n}\sum_{i=0}^{j-3}\sum_{k=0}^{j-3}\varepsilon_j A_{ji}A_{jk}\left(\frac{\kappa_1}{\kappa_2}\right)^{(i)}\left(\frac{\kappa_1}{\kappa_2}\right)^{(k)} = C$$

where C is constant, $j - i \ge 3, \quad j - k \ge 3$.

Example 3.1. $\alpha(t) = \left(\cosh\frac{t}{\sqrt{2}}, \sinh\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$ is a general helix curve in E_1^3 (Figure 1). Tangent vector T makes a constant angle with a fixed direction $W =$

$(0,0,1)$ and also it is clear that α is a unit speed spacelike curve with a timelike principal normal N. The Frenet vectors of α are

$$\begin{aligned} T &= \alpha' = \left(\frac{1}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ N &= \frac{T'}{\kappa} = \left(\cosh\frac{t}{\sqrt{2}}, \sinh\frac{t}{\sqrt{2}}, 0\right), \\ B &= T\times N = \left(\frac{1}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \end{aligned}$$

and the curvature κ_1, the torsion κ_2 of α are

$$\kappa_1 = -\left\langle T', N\right\rangle = \frac{1}{2},\ \kappa_2 = \left\langle N', B\right\rangle = \frac{1}{2}.$$

For $n=3$ the equation (16) is

$$C = \varepsilon_1 + \varepsilon_3 S_3^2 = 1 + \left[\frac{\varepsilon_2\varepsilon_3}{\kappa_2}\kappa_1 S_1\right]^2 = 2.$$

On the other hand from Definition 2.2, $\langle T, W\rangle = \cos\phi = \dfrac{1}{\sqrt{2}}$. Using this result in Corollary 3.1, we get $C = \sec^2\phi = 2$.

Example 3.2. $\alpha(t) = \left(\frac{t}{\sqrt{2}}, \cos\frac{\sqrt{3}t}{\sqrt{2}}, \sin\frac{\sqrt{3}t}{\sqrt{2}}\right)$ is a general helix curve in E_1^3 (Figure 2). The tangent vector T makes a constant angle with a fixed direction $W=(1,0,0)$ and also it is clear that α is a unit speed spacelike curve with a spacelike principal normal N. The Frenet vectors of α are

$$\begin{aligned} T &= \alpha' = \left(\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}\sin\frac{\sqrt{3}t}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\cos\frac{\sqrt{3}t}{\sqrt{2}}\right), \\ N &= \frac{T'}{\kappa} = \left(0, -\cos\frac{\sqrt{3}t}{\sqrt{2}}, -\sin\frac{\sqrt{3}t}{\sqrt{2}}\right), \\ B &= T\times N = \left(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin\frac{\sqrt{3}t}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cos\frac{\sqrt{3}t}{\sqrt{2}}\right) \end{aligned}$$

and the curvature κ_1, the torsion κ_2 of α are

$$\kappa_1 = \left\langle T', N\right\rangle = \frac{3}{2},\ \kappa_2 = -\left\langle N', B\right\rangle = \frac{\sqrt{3}}{2}.$$

For $n = 3$ the equation (16) is

$$C = \varepsilon_1 + \varepsilon_3 S_3^2 = 1 - \left[\frac{\varepsilon_2\varepsilon_3}{\kappa_2}\kappa_1 S_1\right]^2 = -2.$$

On the other hand from Definition 2.2, $\langle T, W\rangle = \sinh\phi = \dfrac{1}{\sqrt{2}}$. Using this result in Corollary 3.1, we get $C = -\varepsilon_1 \csc h^2\phi = -2$.

Example 3.3. $\alpha(t) = \left(\sinh\frac{\sqrt{3}t}{\sqrt{2}}, \cosh\frac{\sqrt{3}t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$ is a general helix curve in E_1^3 (Figure 3). The tangent vector T makes a constant angle with a fixed direction $W = (0, 0, 1)$ and also it is clear that α is a unit speed timelike curve. The Frenet vectors of α are

$$\begin{aligned}
T &= \alpha' = \left(\frac{\sqrt{3}}{\sqrt{2}}\cosh\frac{\sqrt{3}t}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\sinh\frac{\sqrt{3}t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\\
N &= \frac{T'}{\kappa} = \left(\sinh\frac{\sqrt{3}t}{\sqrt{2}}, \cosh\frac{\sqrt{3}t}{\sqrt{2}}, 0\right),\\
B &= T \times N = \left(\frac{1}{\sqrt{2}}\cosh\frac{\sqrt{3}t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sinh\frac{\sqrt{3}t}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)
\end{aligned}$$

and the curvature κ_1, the torsion κ_2 of α are

$$\kappa_1 = \left\langle T', N\right\rangle = \frac{3}{2},\ \kappa_2 = \left\langle N', B\right\rangle = -\frac{\sqrt{3}}{2}.$$

For $n = 3$ the equation (16) is

$$C = \varepsilon_1 + \varepsilon_3 S_3^2 = -1 + \left[\frac{\varepsilon_2\varepsilon_3}{\kappa_2}\kappa_1 S_1\right]^2 = 2.$$

On the other hand from Definition 2.2, $\langle T, W\rangle = \sinh\phi = \dfrac{1}{\sqrt{2}}$. Using this result in Corollary 3.1, we get $C = -\varepsilon_1 \csc h^2\phi = 2$.

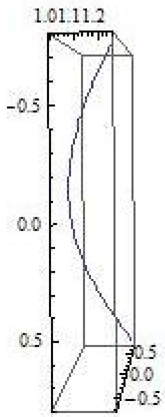

Figure 1

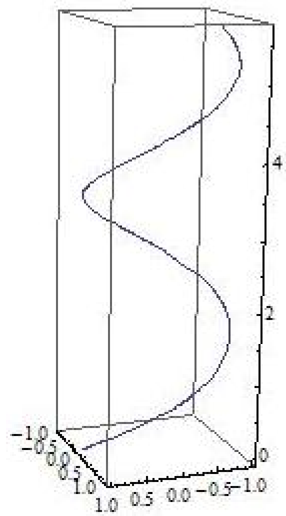

Figure 2

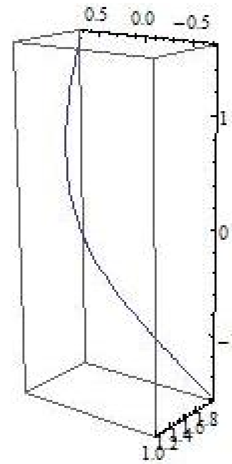

Figure 3

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