A NEW DIFFERENTIAL INEQUALITY

Georgia Irina Oros

Abstract. We find conditions on the complex-valued functions A, B, C, D defined in the unit disc U such that the differential inequality

$$\mathrm{Re}\ [A(z)p^2(z) - B(z)(zp'(z))^2 + C(z)zp'(z) + D(z)] > 0$$

implies $\mathrm{Re}\ p(z) > 0$, where $p \in \mathcal{H}[1, n]$.

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1. Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U],\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U],\ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots,\ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i [1, p. 35].

Lemma A. [1, p. 35] *Let* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ *be a function which satisfies*

$$\text{Re } \psi(\rho i, \sigma; z) \le 0,$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \le -\frac{n}{2}(1+\rho^2)$, $z \in U$ *and* $n \ge 1$.

If $p \in \mathcal{H}[1,n]$ *and*

$$\text{Re } \psi(p(z), zp'(z); z) > 0$$

then

$$\text{Re } p(z) > 0.$$

2. Main results

Theorem 1. *Let* n *be a positive integer. Suppose that unctions* $A, B, C, D : U \to \mathbb{C}$ *satisfy*

$$\begin{cases} i) \quad \text{Re } B(z) \ge 0 \\ ii) \quad \text{Re } A(z) \ge -\text{Re } \left[\frac{n^2}{2}B(z) + \frac{n}{2}C(z)\right] \\ iii) \quad \text{Re } D(z) \le \text{Re } \left[\frac{n^2}{4}B(z) + \frac{n}{2}C(z)\right]. \end{cases} \tag{1}$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\text{Re } [A(z)p^2(z) - B(z)(zp'(z))^2 + C(z)zp'(z) + D(z)] > 0 \tag{2}$$

then

$$\text{Re } p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^2(z) - B(z)(zp'(z))^2 + C(z)zp'(z) + D(z) \tag{3}$$

From (2) we have

$$\text{Re } \psi(p(z), zp'(z); z) > 0, \text{ for } z \in U. \tag{4}$$

For $\sigma, \rho \in \mathbb{R}$ satisfying $\sigma \leq -\frac{n}{2}(1+\rho^2)$, we have

$$-\sigma^2 \leq -\frac{n^2}{4}(1+\rho^2)^2,$$

and by using (1) we obtain:

$$\text{Re } \psi(\rho i, \sigma; z) = \text{Re } [A(z)(\rho i)^2 - B(z)\sigma^2 + C(z)\sigma + D(z)] =$$

$$= -\rho^2 \text{Re } A(z) - \sigma^2 \text{Re } B(z) + \sigma \text{Re } C(z) + \text{Re } D(z) \leq$$

$$\leq -\rho^2 \text{Re } A(z) - \frac{n^2}{4}(1+\rho^2)^2 \text{Re } B(z) - \frac{n}{2}(1+\rho^2)\text{Re } C(z) + \text{Re } D(z) \leq$$

$$\leq -\rho^2 \text{Re } A(z) - \frac{n^2}{4}(1+2\rho^2+\rho^4)\text{Re } B(z) - \frac{n}{2}(1+\rho^2)\text{Re } C(z) + \text{Re } D(z) \leq$$

$$\leq -\frac{n^2}{4}\text{Re } B(z) - \rho^2\left[\text{Re } A(z) + \frac{n^2}{2}\text{Re } B(z) + \frac{n}{2}\text{Re } C(z)\right] -$$

$$-\frac{n^2}{4}\text{Re } B(z) - \frac{n}{2}\text{Re } C(z) + \text{Re } D(z) \leq 0.$$

By using Lemma A we have that $\text{Re } p(z) > 0$. $\square$

If $C(z) \equiv 0$, then Theorem can be rewritten as follows:

Corollary. *Let n be a positive integer. Suppose that the functions $A, B, D : U \to \mathbb{C}$ satisfy*

i) $\text{Re } B(z) \geq 0$

ii) $\text{Re } A(z) \geq -\frac{n^2}{2}\text{Re } B(z)$

iii) $\text{Re } D(z) \leq \frac{n^2}{4}\text{Re } B(z)$.

If $p \in \mathcal{H}[1, n]$ and

$$\text{Re } [A(z)p^2(z) - B(z)(zp'(z))^2 + D(z)] > 0$$

then

$$\text{Re } p(z) > 0.$$

If $A(z) = 1 + z$, $B(z) = 3 + 2z$, $D(z) = \frac{z}{4}$, $n = 1$, then from Corollary 1 we deduce

Example 1. If $p \in \mathcal{H}[1,1]$ then

$$\text{Re }\left[(1+z)p^2(z)-(3+2z)(zp'(z))^2+\frac{z}{4}\right]>0$$

implies

$$\text{Re }p(z)>0.$$

If $A(z)=2+z$, $B(z)=1+\frac{z}{2}$, $D(z)=\frac{z}{2}$, $n=2$, then from Corollary 1 we deduce:

Example 2. If $p \in \mathcal{H}[1,2]$ and

$$\text{Re }\left[(2+z)p^2(z)-\left(1+\frac{z}{2}\right)(zp'(z))^2+\frac{z}{2}\right]>0$$

implies

$$\text{Re }p(z)>0.$$

If $D(z)=\frac{n^2}{4}B(z)+\frac{n}{2}C(z)$, then Theorem can be rewritten as follows:

Corollary 2. *Let n be a positive integer. Suppose that the functions $A, B, C : U \to \mathbb{C}$ satisfy*

i) $\text{Re }B(z)\geq 0, \text{Re }C(z)\geq 0$

ii) $\text{Re }A(z)\geq -\text{Re }\left[\frac{n^2}{2}B(z)+\frac{n}{2}C(z)\right].$

If $p \in \mathcal{H}[1,n]$ and

$$\text{Re }\left[A(z)p^2(z)-B(z)(zp'(z))^2+C(z)zp'(z)+\frac{n^2}{4}B(z)+\frac{n}{2}C(z)\right]>0$$

then

$$\text{Re }p(z)>0.$$

If $A(z)=4+z$, $B(z)=2+z$, $C(z)=3-z$, $n=1$, then from Corollary 2 we deduce

Example 3. If $p \in \mathcal{H}[1,1]$ then

$$\text{Re }\left[(4+z)p^2(z)-(2+z)(zp'(z))^2+(3-z)zp'(z)+2-\frac{z}{4}\right]>0$$

implies

$$\mathrm{Re}\; p(z) > 0.$$

If $A(z) = 1 - z$, $B(z) = 2 - z$, $C(z) = -1 + z$, $n = 2$, then from Corollary 2 we deduce

Example 4. If $p \in \mathcal{H}[1, 2]$ and

$$\mathrm{Re}\; [(1 - z)p^2(z) - (2 - z)(zp'(z))^2 + (-1 + z)zp'(z) + 1] > 0$$

implies

$$\mathrm{Re}\; p(z) > 0.$$

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

Authors:

Georgia Irina Oros
Department of Mathematics
University of Oradea
Oradea
Romania
e-mail:*georgia_oros_ro@yahoo.co.uk*