AN UNIVALENCE CRITERIA FOR A CLASS OF INTEGRAL OPERATORS

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Abstract. In this note it is proved, by the method of subordonation chains, a sufficient condition for the analyticity and the univalence of the functions defined by an integral operator.

1. Introduction

We denote by $U_r = \{ z \in C : |z| < r\}$ the disk of z-plane, where $r \in (0,1]$, $U_1 = U$ and $I = [0,\infty)$. Let A be the class of functions f analytic in U such that $f(0) = 0$, $f'(0) = 1$.

Theorem 1. **([1])** *Let $f \in A$. If for all $z \in U$*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 \tag{1}$$

then the function f is univalent in U.

In order to prove our main result we need the theory of Löewner chains. A function $L : U \times I \longrightarrow C$ is called a Löewner chain if it is analytic and univalent in U and $L(z,s)$ is subordinate to $L(z,t)$, for all $0 \le s \le t < \infty$. Recall that a function $f : U \longrightarrow C$ is said to be subordinate to a function $g : U \to C$ (in symbols $f \prec g$) if there exists a function $w : U \longrightarrow U$ such that $f(z) = g(w(z))$ for all $z \in U$. We also recall the basic result of this theory, from Pommerenke.

Theorem 2. **([2])** *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots,\ a_1(t) \neq 0$ be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and locally uniformly with respect to U_r. For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition $Re\ p(z,t) > 0$, for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

2. Main results

Theorem 3. *Let $f \in A$ and α be a complex number, $Re\ \alpha > 0$. If the following inequalities*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 \tag{2}$$

and

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right)|z|^{4\alpha} + \frac{1-|z|^{4\alpha}}{2\alpha}\left[2\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) - \alpha\right] + \right. \tag{3}$$

$$\left.\frac{(1-|z|^{4\alpha})^2}{4\alpha^2|z|^{4\alpha}}\left[\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) + (1-\alpha)\left(\frac{f(z)}{z} - 1\right)\right]\right| \leq 1$$

are true for all $z \in U \setminus \{0\}$, then the function F_α,

$$F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u)du\right)^{1/\alpha} \tag{4}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. Let us prove that there exists a real number $r \in (0,1]$ such that the function $L(z,t) : U_r \times I \longrightarrow C$, defined formally by

$$L(z,t) = \left[2\alpha \int_0^{e^{-t}z} u^{\alpha-1} f'(u)du + \frac{(e^{4\alpha t}-1)e^{(2-\alpha)t} z^{\alpha-2} f^2(e^{-t}z)}{1 - \frac{e^{4\alpha t}-1}{2\alpha}\left(\frac{f(e^{-t}z)}{e^{-t}z} - 1\right)}\right]^{1/\alpha} \tag{5}$$

is analytic in U_r, for all $t \in I$.
Because $f \in A$, it is easy to see that the function

$$g_1(z,t) = 2\alpha \int_0^{e^{-t}z} u^{\alpha-1} f'(u)du\ ,$$

can be written as $g_1(z,t) = z^\alpha \cdot g_2(z,t)$, where $g_2(z,t)$ is analytic in U, for all $t \in I,\ g_2(0,t) = 2e^{-\alpha t}$. For all $t \in I$ and $z \in U$, the function

$$g_3(z,t) = 1 - \frac{e^{4\alpha t}-1}{2\alpha}\left(\frac{f(e^{-t}z)}{e^{-t}z} - 1\right)$$

is analytic in U and $g_3(0,t)=1$. Then there is a disk $U_{r_1},\ 0<r_1<1$ in which $g_3(z,t)\neq 0$, for all $t\in I$. It follows that the function g_4 is also analytic in U_{r_1}, where

$$g_4(z,t)=g_2(z,t)+\frac{(e^{3\alpha t}-e^{-\alpha t})\left(\frac{f(e^{-t}z)}{e^{-t}z}\right)^2}{g_3(z,t)}$$

and $g_4(0,t)=e^{3\alpha t}+e^{-\alpha t}$. From $Re\alpha>0$ we deduce that $g_4(0,t)\neq 0$ for all $t\in I$. Therefore, there is a disk $U_{r_2},\ 0<r_2\leq r_1$ in which $g_4(z,t)\neq 0$, for all $t\in I$ and we can choose an analytic branch of $[g_4(z,t)]^{1/\alpha}$, denoted by $g(z,t)$. We choose the uniform branch which is equal to $a_1(t)=(e^{3t\alpha}+e^{-\alpha t})^{1/\alpha}$ at the origin, and for $a_1(t)$ we fix a determination.

From these considerations it follows that the relation (5) may be written as

$$L(z,t)=z\cdot g(z,t)=a_1(t)z+a_2(t)z^2+\dots$$

and is analytic in U_{r_2}, for all $t\in I$, $a_1(t)=e^{3t}(1+e^{-4\alpha t})^{1/\alpha}$. We have $\lim_{t\to\infty}|a_1(t)|=\infty$ and $a_1(t)\neq 0$.

From the analyticity of $L(z,t)$ in U_{r_2}, it follows that there is a number $r_3,\ 0<r_3<r_2$, and a constant $K=K(r_3)$ such that

$$\mid L(z,t)/e^t\mid<K, \qquad \forall z\in U_{r_3},\quad t\in I,$$

and then $\{L(z,t)/e^t\}$ is a normal family in U_{r_3}. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T>0$ and $r_4,\ 0<r_4<r_3$, there exists a constant $K_1>0$ (that depends on T and r_4) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right|<K_1, \qquad \forall z\in U_{r_4},\quad t\in[0,T].$$

It follows that the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_4}. We also have that the function

$$p(z,t)=z\frac{\partial L(z,t)}{\partial z}\Bigg/\frac{\partial L(z,t)}{\partial t}$$

is analytic in U_r, $0<r<r_4$, for all $t\in I$.

In order to prove that the function $p(z,t)$ has an analytic extension, with positive real part in U, for all $t\in I$, it is sufficient to show that the function

$w(z,t)$ defined in U_r by

$$w(z,t)=\frac{p(z,t)-1}{p(z,t)+1}$$

can be continued analytically in U and that $|w(z,t)|<1$ for all $z\in U$ and $t\in I$.
By simple calculations, we obtain

$$w(z,t)=\left(\frac{e^{-2t}z^2f'(e^{-t}z)}{f^2(e^{-t}z)}-1\right)e^{-4\alpha t}+\frac{1-e^{-4\alpha t}}{2\alpha}\left[\left(2\frac{e^{-2t}z^2f'(e^{-t}z)}{f^2(e^{-t}z)}-1\right)-\alpha\right]$$

$$+\frac{(1-e^{-4\alpha t})^2}{4\alpha^2e^{-4\alpha t}}\left[\left(\frac{e^{-2t}z^2f'(e^{-t}z)}{f^2(e^{-t}z)}-1\right)+(1-\alpha)\left(\frac{f(e^{-t}z)}{e^{-t}z}-1\right)\right] \quad (6)$$

From (2) and (3) we deduce that the function $w(z,t)$ is analytic in the unit disk and

$$|\, w(z,0)\,|=\left|\frac{z^2f'(z)}{f^2(z)}-1\right|<1 \quad (7)$$

We observe that $w(0,t)=0$. Let t be a fixed number, $t>0$, $z\in U$, $z\neq 0$. Since $|e^{-t}z|\leq e^{-t}<1$ for all $z\in\overline{U}=\{z\in C:|z|\leq 1\}$ we conclude that the function $w(z,t)$ is analytic in $\overline{U}$. Using the maximum modulus principle it follows that for each $t>0$, arbitrary fixed, there exists $\theta=\theta(t)\in R$ such that

$$|w(z,t)|<\max_{|\xi|=1}|w(\xi,t)|=|w(e^{i\theta},t)|, \quad (8)$$

We denote $u=e^{-t}\cdot e^{i\theta}$. Then $|u|=e^{-t}<1$ and from (6) we get

$$w(e^{i\theta},t)=\left(\frac{u^2f'(u)}{f^2(u)}-1\right)|u|^{4\alpha}+\frac{1-|u|^{4\alpha}}{2\alpha}\left[2\left(\frac{u^2f'(u)}{f^2(u)}-1\right)-\alpha\right]+$$

$$\frac{(1-|u|^{4\alpha})^2}{4\alpha^2|u|^{4\alpha}}\left[\left(\frac{u^2f'(u)}{f^2(u)}-1\right)+(1-\alpha)\left(\frac{f(u)}{u}-1\right)\right].$$

Since $u\in U$, the inequality (3) implies $|w(e^{i\theta},t)|\leq 1$ and from (7) and (8) we conclude that $|w(z,t)|<1$ for all $z\in U$ and $t\geq 0$.

From Theorem 2.1 it results that the function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t \in I$. For $t=0$ we conclude that the function

$$L(z,0) = \left(2\alpha \int_0^z u^{\alpha-1} f'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U, and then the function defined by (4)

$$F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u)du\right)^{1/\alpha}$$

is analytic and univalent in U.

Remark 1. The condition (2) of the Theorem 2.1 which is just Ozaki-Nunokawa's univalence criterion, assures the univalence of the function f. For $\alpha = 1/2$ we get

Corollary 1. *Let $f \in A$. If the following inequalities*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 \tag{9}$$

and

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right)\frac{1}{|z|^2} - \frac{1-|z|^2}{2} + \frac{(1-|z|^2)^2}{2|z|^2}\left(\frac{f(z)}{z} - 1\right)\right| \le 1 \tag{10}$$

are true for all $z \in U \setminus \{0\}$, then the function

$$F(z) = \left(\int_0^z \frac{f'(u)}{2\sqrt{u}}du\right)^2 \tag{11}$$

is analytic and univalent in U.

Example 1. *Let the function*

$$f(z) = \frac{z}{1 - \frac{z^2}{4}} \tag{12}$$

Then f is univalent in U and the function F defined by (11)is analytic and univalent in U.

Proof. We have
$$\frac{z^2 f'(z)}{f^2(z)} - 1 = \frac{z^2}{4} \tag{13}$$
and
$$\frac{f(z)}{z} - 1 = \frac{z^2}{4 - z^2} \tag{14}$$
It is clear that the condition (2) of the Theorem 3.1 is satisfied, and then the function f is univalent in U.
Taking into account (13) and (14), the condition (10) of the Corollary 1 becomes
$$\left| \frac{z^2}{4} \cdot \frac{1}{|z|^2} - \frac{1 - |z|^2}{2} + \frac{(1 - |z|^2)^2}{2|z|^2} \left(\frac{z^2}{4 - z^2} \right) \right| \leq$$
$$\frac{1}{4} + \frac{1 - |z|^2}{2} + \frac{(1 - |z|^2)^2}{6} = \frac{1}{12}[2|z|^4 - 10|z|^2 + 11] < 1$$
because the greatest value of the function $g(x) = 2x^2 - 10x + 11$, for $x \in [0, 1]$ is taken for $x = 0$ and is $g(0) = 11$. Therefore the function F defined by (11) is analytic and univalent in U.

References

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