ON THE UNIVALENCE OF SOME INTEGRAL OPERATORS

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Abstract. In this work is considered the class of univalent functions defined by the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1$, $|z| < 1$, where $f(z) = z + a_2 z^2 + \dots$ is analytic in the open unit disk $U = \{z \in C \,|\, |z| < 1\}$. In view of some integral operators $H_{\alpha,\beta}$, G_α and L_γ, sufficient conditions for univalence of the integral operators are discussed.

2000 Mathematics Subject Classification: 30C45

Keywords and phrases: Integral operator, Univalence

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U = \{\, z \in C \,|\, |z| < 1\}$. We denote by S the class of the functions $f \in A$ which are univalent in U.

For $f \in A$, the integral operator $H_{\alpha,\beta}$ is defined by

$$H_{\alpha,\beta} f(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}} \tag{1.2}$$

for some complex number α and $\beta \, (\alpha \neq 0, \beta \neq 0)$.

Also, the integral operator G_α is given by

$$G_\alpha f(z) = \left[\alpha \int_0^z (f(u))^{\alpha-1} \, du\right]^{\frac{1}{\alpha}} \tag{1.3}$$

for some complex number $\alpha \, (\alpha \neq 0)$, $f \in A$.

For $f \in A$, the integral operator L_γ is defined by

$$L_\gamma f(z) = \left[\gamma \int_0^z u^{2\gamma-2} \left(e^{f(u)}\right)^{\gamma-1} du\right]^{\frac{1}{\gamma}} \tag{1.4}$$

for some complex number γ, $(\gamma \neq 0)$.

In the present paper, we consider some sufficient conditions for the integral operators to be in the class S.

2. Univalence of the integral operators

In order to discuss our problems for univalence of the integral operators, we have to recall here the following lemmas.

Lemma 2.1.[3]. *Assume that the $f \in A$ satisfies the condition*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1,\ z \in U \tag{2.1}$$

then f is univalent in U.

Lemma 2.2.[4]. *If $f \in A$ satisfies*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,\ z \in U \tag{2.2}$$

for some complex number α with $Re\alpha > 0$, then the integral operator F_β defined by

$$F_\beta f(z) = \left\{\beta \int_0^z u^{\beta-1} f'(u) du\right\}^{\frac{1}{\beta}} \tag{2.3}$$

is in the class S for a complex number β such that $Re\beta \geq Re\alpha$.

Schwartz Lemma [1]. *Let f the function regular in the disk $U_R = \{z \in C:\ |z| < R\}$, with $|f(z)| < M, z \in U_R$, and M fixed. If f has in $z = 0$ one zero multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ z \in U_R \tag{2.4}$$

the equality (in the inequality (2.4) for $z \neq 0$) can hold only if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

Now we derive

Theorem 2.1. *Let $f \in A$ satisfy (2.1), α be a complex number, $Re\alpha > 0, M$ be a real number and $M > 1$.*

If

$$|f(z)| < M,\ z \in U \tag{2.5}$$

and

$$|\alpha| Re\alpha \geq 2M + 1,\ \textit{for } Re\alpha \in (0, 1) \tag{2.6}$$

or

$$|\alpha| \geq 2M + 1,\ \textit{for } Re\alpha \in [1, \infty) \tag{2.7}$$

then for complex number β such that $Re\beta \geq Re\alpha$, the integral operator $H_{\alpha,\beta}$ given by (1.2) is in the class S.

Proof. Let us define the function g by

$$g(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du. \tag{2.8}$$

The function g is regular in U. We have

$$g'(z) = \left(\frac{f(z)}{z}\right)^{\frac{1}{\alpha}},\ g''(z) = \frac{1}{\alpha}\left(\frac{f(z)}{z}\right)^{\frac{1}{\alpha}-1} \frac{zf'(z) - f(z)}{z^2}$$

and

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left|\frac{zg''(z)}{g'(z)}\right| = \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \frac{1}{|\alpha|} \left|\frac{zf'(z)}{f(z)} - 1\right| \tag{2.9}$$

for all $z \in U$. From (2.9) we obtain

$$\frac{1 - |z|^{2Re\alpha}}{|\alpha| Re\alpha} \left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1 - |z|^{2Re\alpha}}{|\alpha| Re\alpha} \left(\left|\frac{z^2 f'(z)}{f^2(z)}\right| \frac{|f(z)|}{|z|} + 1\right) \tag{2.10}$$

for all $z \in U$.

By the Schwarz Lemma also $|f(z)| \leq M|z|, z \in U$ and using (2.10) we get

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1 - |z|^{2Re\alpha}}{|\alpha| Re\alpha} \left(\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| M + M + 1\right). \tag{2.11}$$

Since f satisfies the condition (2.1) then from (2.11) we have

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha}\left(2M+1\right). \tag{2.12}$$

For $Re\alpha \in (0,1)$ we have $1-|z|^{2Re\alpha} \leq 1-|z|^2$, $z \in U$ and from (2.12), (2.6) we obtain that

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1 \tag{2.13}$$

for all $z \in U$.

For $Re\alpha \in [1,\infty)$ we have $\frac{1-|z|^{2Re\alpha}}{Re\alpha} \leq 1-|z|^2$, $z \in U$ and from (2.12), (2.7) we get

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1 \tag{2.14}$$

for all $z \in U$.

Consequently, in view of Lemma 2.2, we prove that $H_{\alpha,\beta}f(z) \in S$.

Theorem 2.2. *Let $f \in A$ satisfy (2.1), α be a complex number with $Re\alpha > 0$, M be a real number and $M > 1$.*
If

$$|f(z)| < M,\ z \in U \tag{2.15}$$

and

$$\frac{|\alpha-1|}{Re\alpha} \leq \frac{1}{2M+1},\ \textit{for } Re\alpha \in (0,1) \tag{2.16}$$

or

$$|\alpha-1| \leq \frac{1}{2M+1},\ \textit{for } Re\alpha \in [1,\infty) \tag{2.17}$$

then the integral operator G_α given by (1.3) is in the class S.

Proof. From (1.3) we have

$$G_\alpha f(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(\frac{f(u)}{u}\right)^{\alpha-1} du\right]^{\frac{1}{\alpha}}. \tag{2.18}$$

Let us consider the function

$$p(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha-1} du. \tag{2.19}$$

The function p is regular in U. From (2.19) we get

$$p'(z)=\left(\frac{f(z)}{z}\right)^{\alpha-1},\ p''(z)=(\alpha-1)\left(\frac{f(z)}{z}\right)^{\alpha-2}\frac{zf'(z)-f(z)}{z^2}.$$

We have

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zp''(z)}{p'(z)}\right|\le\frac{1-|z|^{2Re\alpha}}{Re\alpha}|\alpha-1|\left(\left|\frac{zf'(z)}{f(z)}\right|+1\right). \tag{2.20}$$

Hence, we obtain

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zp''(z)}{p'(z)}\right|\le|\alpha-1|\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left(\left|\frac{z^2f'(z)}{f^2(z)}\right|\left|\frac{f(z)}{z}\right|+1\right) \tag{2.21}$$

for all $z\in U$.

Applying Schwarz Lemma we have $|f(z)|\le M|z|$, $z\in U$ and using (2.21) we obtain

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zp''(z)}{p'(z)}\right|\le|\alpha-1|\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left(\left|\frac{z^2f'(z)}{f^2(z)}-1\right|M+M+1\right). \tag{2.22}$$

By the condition (2.1) for f, we get

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zp''(z)}{p'(z)}\right|\le\frac{1-|z|^{2Re\alpha}}{Re\alpha}|\alpha-1|\left(2M+1\right). \tag{2.23}$$

For $Re\alpha\in(0,1)$ we have $1-|z|^{2Re\alpha}\le1-|z|^2$ and from (2.23), (2.16) we obtain that

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zp''(z)}{p'(z)}\right|\le1 \tag{2.24}$$

for all $z\in U$.

For $Re\alpha\in[1,\infty)$ we have $\frac{1-|z|^{2Re\alpha}}{Re\alpha}\le1-|z|^2$ and from (2.23), (2.17) we get

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zp''(z)}{p'(z)}\right|\le1 \tag{2.25}$$

for all $z \in U$.

Now (2.24), (2.25) and Lemma 2.2 for $\beta = \alpha$, imply that $G_\alpha f(z) \in S$.

Theorem 2.3. *Let $f \in A$ satisfy (2.1), α, γ be complex numbers, $Re\gamma \geq Re\alpha > 0$, M be a real number and $M > 1$.*

If

$$|f(z)| < M, z \in U \tag{2.26}$$

and

$$\frac{|\gamma - 1|}{Re\gamma} \leq \frac{54M^4}{(12M^4+1)\sqrt{12M^4+1}+36M^4-1}, \quad \textit{for } Re\gamma \in (0,1) \tag{2.27}$$

or

$$|\gamma - 1| \leq \frac{54M^4}{(12M^4+1)\sqrt{12M^4+1}+36M^4-1}, \quad \textit{for } Re\gamma \in [1,\infty) \tag{2.28}$$

then the integral operator L_γ given by (1.4) is in the class S.

Proof. We observe that

$$L_\gamma f(z) = \left[\gamma \int_0^z u^{\gamma-1}\left(ue^{f(u)}\right)^{\gamma-1} du\right]^{\frac{1}{\gamma}}. \tag{2.29}$$

Let us define the function g is

$$g(z) = \int_0^z \left(ue^{f(u)}\right)^{\gamma-1} du. \tag{2.30}$$

The function g is regular in U.

From (2.30) we have

$$\frac{g''(z)}{g'(z)} = (\gamma - 1)\frac{zf'(z)+1}{z} \tag{2.31}$$

and hence we get

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq |\gamma - 1|\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left(\left|\frac{z^2f'(z)}{f^2(z)}\right|\frac{|f^2(z)|}{|z|}+1\right) \tag{2.32}$$

for all $z \in U$.

By the Schwarz Lemma also $|f(z)| \leq M|z|, z \in U$ and using (2.32) we obtain

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq |\gamma-1|\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left(\left|\frac{z^2f'(z)}{f^2(z)}-1\right|M^2|z|+M^2|z|+1\right). \tag{2.33}$$

for all $z \in U$.

Since f satisfies the condition (2.1) then from (2.33) we have

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq |\gamma-1|\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left(2M^2|z|+1\right). \tag{2.34}$$

for all $z \in U$.

For $Re\gamma \in (0,1)$ we obtain $1-|z|^{2Re\gamma} \leq 1-|z|^2$, $z \in U$ and from (2.34) we get

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{|\gamma-1|}{Re\gamma}(1-|z|^2)(2M^2|z|+1) \tag{2.35}$$

for all $z \in U$.

Let us consider the function $Q:[0,1] \to \Re$, $Q(x)=(1-x^2)(2M^2x+1)$, $x=|z|$.

We have

$$Q(x) \leq \frac{(12M^4+1)\sqrt{12M^4+1}+36M^4-1}{54M^4} \tag{2.36}$$

for all $x \in [0,1]$.

From (2.27), (2.36) and (2.35) we conclude that

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1,\ z \in U,\ Re\gamma \in (0,1) \tag{2.37}$$

For $Re\gamma \in [1,\infty)$ we have $\frac{1-|z|^{2Re\gamma}}{Re\gamma} \leq 1-|z|^2$, $z \in U$ and from (2.34) we obtain

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq |\gamma-1|(1-|z|^2)(2M^2|z|+1). \tag{2.38}$$

From (2.28),(2.36) and (2.38) we have

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1,\ z \in U,\ Re\gamma \in [1,\infty). \tag{2.39}$$

Now (2.37), (2.39) and Lemma 2.2 for $\beta = \gamma$ imply that the integral operator L_γ define by (1.4) is in the class S.

Remark. For $0 < M \leq 1$, Theorem 2.1, Theorem 2.2 and Theorem 2.3 hold only in the case $f(z) = Kz$, where $|K| = 1$.

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