A GENERAL CLASS OF NONLINEAR UNIVARIATE SUBDIVISION ALGORITHMS AND THEIR C^2 SMOOTHNESS

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Abstract. Considering subdivision schemes in nonlinear geometries, it is natural to define an analogues subdivision of a linear one in terms of some functions and new variables reflecting the geometry. In the present work we introduce the notion of analogues of a given linear subdivision in a general setting such that some important examples like log-exponential subdivision in Lie groups (also infinite dimensional applications) are covered. Moreover, we prove that C^2 smoothness of the linear subdivision implies the same property for the analogues one. Moreover, we present some applications to Lie groups and symmetric spaces.

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1.Introduction

In view of the wide range of applications of subdivision processes in nonlinear geometries there has been a growing attention to this topic in the recent years. Many important such applications arrise in continum mechanics by consideration of data corresponding to strain or stress tensor, or in elasticity (deformation tensors) as well as medical imaging (diffusion tensors). Further examples are provided by situations when data naturally live in some Lie groups, for instance vehicle headings or motion of rigid body. For a comprehensive study of many important examples we refer to [5].

Considering subdivision schemes in nonlinear geoemetries, it is natural to define an analogues subsdivision of a linear one in terms of new variables reflecting the geometry. For instance the notion of subdivision in manifolds by means of geodesic averaging as well as log-exponential subdivision in Lie groups and their smoothness properties using the general methods introduced in [7] and [6] has been established in [7], [9] and [8]. For further readings on smoothness of subdivision of manifold- or Lie group-valued data we also refer to [10] and [2].

Definition of the mentioned analogue subdivision schemes uses tangent vectors and the Riemannian exponential map or resp. the exponential map of the Lie group. An important task is then, to understand the relation between convergence and smoothness of the linear subdivision and the analogues nonlinear one.

In the present work we introduce the notion of analogues of a given linear subdivision in a general setting such that some important examples like log-exponential subdivision in Lie groups (also infinite dimensional applications) are covered. Moreover, we prove that C^2 smoothness of the linear subdivision implies the same property for the analogues one, provided the linear subdivision scheme and its difference schemes up to order 3 satisfy certain boundedness conditions.

2.Analogues of a subdivision

Let $(E, \| \,\|)$ ba a Banach space, $N > 1$ an integer and $a := \{a_i\}_{i\in\mathbb{Z}}$ a finite sequence in E with

$$\sum_{j\in\mathbb{Z}} a_{l-jN} = 1,\; l = 0, \dots, N-1. \tag{1}$$

We denote the linear subdivision associated to the sequence a by S, i.e.

$$Sp_{l+iN} = \sum_{j\in\mathbb{Z}} a_{l-jN} p_{i+j} \tag{2}$$

for a sequence of points $p := \{p_i\}_{i\in\mathbb{Z}}$ (control points). The number of rules N is called dialtion factor, the sequence a mask and the Laurent polynomial $a(z) := \sum_{i\in\mathbb{Z}} a_i z^i$ associated to a the symbol of S. Condition (1) ensures convergence of Sp for all p and is called affine invariance.

Suppose that $f : M \times E \to M$ is C^2 with the following (local) properties: each $x \in M$ has a neighbourhood U such that

$$f(x, 0) = x, \tag{3}$$

$$D_1 f(x, 0) = Id_E,\; D_2 f(x, 0)^{-1} \text{ exist}, \tag{4}$$

$$D_{11} f(x, 0) = 0,\; D_{12} f(x, 0)(D_2 f(x, 0)v, w) = D_{22} f(x, 0)(v, w) \text{ for all } v, w \in E. \tag{5}$$

Particularly, property (4) implies that $f(x, .)$ is a local diffeomorphism around $(x, 0)$.

Now, suppose that the points p_i are close enough in the following sence. $p \subset U \subset M$ such that $f_{|U}$ enjoys the above properties. For simplicity we denote the restriction of f to U by f. Then the equations

$$p_{i+1} = f(p_i, v_i) \tag{6}$$

define a sequence of vectors $v := \{v_i\}_{i \in \mathbb{Z}}$ in E. Now we define the f-analogue of S as

$$Tp_{l+iN} = f(p_i, \sum_{j \in \mathbb{Z}} a_{l-jN}(v_{i+j-1} + \cdots + v_i)),\ l = 0, \ldots, N-1. \tag{7}$$

Note that (7) can be written as

$$Tp_{l+iN} = f(p_i, \sum_{k \in \mathbb{Z}} (\sum_{j-1 \geq k} a_{l-jN}) v_{i+k},\ l = 0, \ldots, N-1. \tag{8}$$

We also denote that by (3)

$$f(f(\ldots(f(x,0),0),\ldots,0) = x. \tag{9}$$

As all considerations in the present work are local, we may and do assume $U \subset E$.

3.Smoothness

In the next two subsections we summarize some notions and facts which will used to prove smoothness. For the corresponding background material we refer to [6]. We denote the forward difference operator Δ operating on the vector space of sequences in E by Δ, i.e.,

$$\Delta^{i+1}p := \Delta(\Delta^i p),\ \Delta^0 p := p.$$

3.1.Derived schemes and proximity

Derived (also called difference) schemes of the subdivision scheme S are defined by

$$S_{i+1}\Delta p := N\Delta S_i p,\ S_0 := S.$$

By affine invariance the first derived scheme S_1 always exist. Derived schemes up to order k exist if and only if $a(z)$ is divisible by $(1 + z + \cdots + z^{N-1})^k$. We denote $\mu_k := \frac{1}{N}\|S_k\|$.

To prove smoothness properties of the analogue subdivision scheme we use the method of the so-called proximity conditions. We first need the following technical lemmas.

Lemma 1. *Suppose that derived schems of S up to order n exist. Denote*

$$c_l^k := (\sum_{j\in\mathbb{Z}} j a_{l-jN})^k - \sum_{j\in\mathbb{Z}} j^k a_{l-jN},\ l = 0, \dots, N-1. \tag{10}$$

Then the following holds

$$\Delta^j c_l^{j+1} = 0 \text{ for all } l = 0, \dots, N-1 \text{ and } j = 1, \dots, n-1. \tag{11}$$

Proof. Denoting $\xi := e^{2\pi i/N}$, existence of derived schemes up to order n is equivalent to

$$\sum_{l=0}^{N-1}\sum_{j\in\mathbb{Z}}(l+jN)(l-1+jN)\dots(l-n+1+jN)a_{l+jN}\xi^{k(l-n)} = 0$$

for $k = 1, \dots, N-1$. These equations can be written as

$$\sum_{l=0}^{N-1}\sum_{j\in\mathbb{Z}}(l+jN)(l-1+jN)\dots(l-n+1+jN)a_{l+jN}\xi^{kl} = 0.$$

with $k = 1, \dots, N-1$. Defining

$$\alpha_{n,l} := \sum_{j\in\mathbb{Z}}(l+jN)(l-1+jN)\dots(l-n+1+jN)a_{l+jN},$$

we see that $\alpha_{n,l}$ is independent of l and in view of

$$\sum_{l=0}^{N-1}\alpha_{n,l} = a^{(n)}(1)$$

we have $\alpha_{n,l} = \frac{a^{(n)}(1)}{N}$. Denoting $b_{n,l} := \sum_{j\in\mathbb{Z}} j^n a_{l+jN}$ and $\sum_{i=0}^{n}\sigma_{i,l}z^i := (l+z)(l-1+z)\dots(l-n+1+z)$ it follows that $b_{n,l}$ is given by the recursion formula

$$b_{n,l} = \frac{1}{N^{n+1}}(a^{(n)}(1) - \sum_{i=0}^{n-1}\sigma_{i,l}N^{i+1}b_{i,l}).$$

Hence $b_{n,l}$ and $b^n_{1,l}$ are polynomials in l of degree n with equal coefficients of $(n-1)$-th resp. n-th monomial (given by $(\frac{-1}{N})^{n-1}na'(1)$ resp. $(\frac{-1}{N})^n$). Therefore $\Delta^n c^{n+1}_l = (-1)^{n+1}\Delta^n(b^n_{1,l} - b_{n,l}) = 0$. □

Lemma 2. *There are positive constants C and C' such that*

$$\|v\| \leq C\|\Delta p\| \tag{12}$$
$$\|\Delta v\| \leq C'(\|\Delta^2 p\| + \|\Delta p\|^2) \tag{13}$$

Proof. Since $f(p_i, .)$ is a local deiffeomorphism, there is $C > 0$ such that

$$C\|v_i\| \leq \|f(p_i, v_i) - f(p_i, 0)\| = \|p_{i+1} - p_i\| = \|\Delta p_i\|$$

holds. This proves the first inequality.
Moreover, linearization of $\Delta^2 p_i$ at $v = 0$ gives

$$\begin{aligned}\Delta^2 p_i &= f(f(p_i, v_i), v_{i+1}) - 2f(p_i, v_i) + p_i\\ &= D_2 f(p_i, 0)\Delta v_i + o(\|v_i\|^2).\end{aligned}$$

Using (12) we get the inequality

$$\|\Delta v_i\| \leq \|D_2 f(p_i, 0)^{-1}\|(\|\Delta^2 p_i\| + o(\|v\|^2))\|D_2 f(p_i, 0)^{-1}\|(\|\Delta^2 p_i\| + C\|\Delta p\|^2)$$

from which (13) immediately follows. □

Lemma 3. *(i) There is a positive constant C such that*

$$\|Sp - Tp\| \leq C\|\Delta p\|^2 \tag{14}$$

(ii) Suppose that S_2 and S_3 exist. Then there exist a positive constant C' such that

$$\|Sp - Tp\| \leq C'(\|\Delta p\|\|\Delta^2 p\| + \|\Delta p\|^3) \tag{15}$$

Proof. A straighforward calculation shows that

$$\partial_{v_k} p_j = \begin{cases} D_1 f(p_{j-1}, v_{j-1}) \dots D_1 f(p_{k+1}, v_{k+1}) D_2 f(p_k, v_k) & \text{for } k = 0, \dots, j-1.\\ 0 & \text{else}\end{cases} \tag{16}$$

Ad i) For $l = 0, \ldots, N-2$ the linearization of ΔSp_{l+iN} at $v = 0$ reads as

$$\Delta(Tp - Sp)_{l+iN} \overset{(1)}{=} \sum_{j\in\mathbb{Z}} \sum_{k\leq j-1} a_{l+1-jN} D_2 f(p_i, 0) v_{i+k}$$
$$- \sum_{j\in\mathbb{Z}} \sum_{k\leq j-1} a_{l-jN} D_2 f(p_i, 0) v_{i+k}$$
$$- \sum_{j\in\mathbb{Z}} \sum_{k\leq j-1} a_{l+1-jN} (D_1 f(p_i, 0))^{j-k-1} D_2 f(p_i, 0) v_{i+k}$$
$$+ \sum_{j\in\mathbb{Z}} \sum_{k\leq j-1} a_{l-jN} (D_1 f(p_i, 0))^{j-k-1} D_2 f(p_i, 0) v_{i+k}$$

For $l = N-1$ we have

$$\Delta(Tp - Sp)_{l+iN} \overset{(1)}{=} D_1 f(p_i, 0) D_2 f(p_i, 0) v_i + \sum_{j\in\mathbb{Z}} \sum_{k\leq j-1} a_{-jN} D_2 f(p_i, 0) v_{i+1+k}$$
$$- \sum_{j\in\mathbb{Z}} \sum_{k\leq j-1} a_{-1+(1-j)N} D_2 f(p_i, 0) v_{i+k}$$
$$- \sum_{j\in\mathbb{Z}} \sum_{k\leq j} a_{-jN} (D_1 f(p_i, 0))^{j-k} D_2 f(p_i, 0) v_{i+k}$$
$$+ \sum_{j\in\mathbb{Z}} \sum_{k\leq j-1} a_{-1+(1-j)N} (D_1 f(p_i, 0))^{j-k-1} D_2 f(p_i, 0) v_{i+k}$$

Using $D_1 f(p_i, 0) = Id$ and affine invariance we arrive at

$$\Delta(Tp - Sp) \overset{(1)}{=} 0$$

from which the desired inequality follows.
Ad ii) In view of (16) we have $\partial_{v_{i+k'}} \partial_{v_{i+k'}} p_{i+j} = 0$ for $k \notin \{0, \ldots, j-1\}$ or $k' \notin \{0, \ldots, j-1\}$. Furthermore for $k, k' = 0, \ldots, j-1$

$$\partial_{v_{i+k'}} \partial_{v_{i+k}} p_{i+j}$$
$$= \partial_{v_{i+k'}} D_1 f(p_{i+j-1}, v_{i+j-1}) D_1 f(p_{i+j-2}, v_{i+j-2}) \ldots D_1 f(p_{i+k+1}, v_{i+k+1}) D_2 f(p_{i+k}, v_{i+k})$$
$$+ D_1 f(p_{i+j-1}, v_{i+j-1}) \partial_{v_{i+k'}} D_1 f(p_{i+j-2}, v_{i+j-2}) \ldots D_1 f(p_{i+k+1}, v_{i+k+1}) D_2 f(p_{i+k}, v_{i+k})$$
$$\vdots$$
$$+ D_1 f(p_{i+j-1}, v_{i+j-1}) \ldots \partial_{v_{i+k'}} D_1 f(p_{i+k+1}, v_{i+k+1}) D_2 f(p_{i+k}, v_{i+k})$$
$$+ D_1 f(p_{i+j-1}, v_{i+j-1}) \ldots D_1 f(p_{i+k+1}, v_{i+k+1}) \partial_{v_{i+k'}} D_2 f(p_{i+k}, v_{i+k})$$

Therefore we have

$$2Sp_{l+iN} \overset{(2)}{=} \sum_{j\in\mathbb{Z}} a_{l-jN}[\sum_{k\le j-1}(\sum_{k'=k}((j-k-1)D_{11}f(p_i,0)(D_2f(p_i,0)v_{i+k},D_2f(p_i,0)v_{i+k'})$$
$$+D_{22}f(p_i,0)(v_{i+k},v_{i+k'}))$$
$$+\sum_{k'<k}((j-k-1)D_{11}f(p_i,0)(D_2f(p_i,0)v_{i+k},D_2f(p_i,0)v_{i+k'})$$
$$+D_{12}f(p_i,0)(D_2f(p_i,0)v_{i+k'},v_{i+k}))$$
$$+\sum_{k'>k}((j-k'-1)D_{11}f(p_i,0)(D_2f(p_i,0)v_{i+k'},D_2f(p_i,0)v_{i+k})$$
$$+D_{12}f(p_i,0)(D_2f(p_i,0)v_{i+k},v_{i+k'}))]$$

Using $D_{11}f(p_i,0)=0$ and $D_{12}f(p_i,0)\circ D_2f(p_i,0)=D_{22}f(p_i,0)$ we arrive at

$$2Sp_{l+iN} \overset{(2)}{=} \sum_{j\in\mathbb{Z}} a_{l-jN} \sum_{k',k\le j-1} D_{22}f(p_i,0)(v_{i+k},v_{i+k'}),\ \ l=0,\dots,N-1.$$

For $l=N-1$ we may write

$$2Sp_{l+1+iN}=2\sum_{j\in\mathbb{Z}} a_{-jN}p_{i+j+1} \overset{(2)}{=} \sum_{j\in\mathbb{Z}} a_{-jN} \sum_{k',k\le j} D_{22}f(p_i,0)(v_{i+k},v_{i+k'})$$
$$=\sum_{j\in\mathbb{Z}} a_{-jN}(\sum_{k',k\le j-1} D_{22}f(p_i,0)(v_{i+1+k},v_{i+1+k'})+\sum_{k\le j-1} D_{22}f(p_i,0)(v_{i+k+1},v_i)$$
$$+\sum_{k'\le j-1} D_{22}f(p_i,0)(v_i,v_{i+k'+1})+D_{22}f(p_i,0)(v_i,v_i))$$
$$=\sum_{j\in\mathbb{Z}} a_{N-jN}\sum_{k',k\le j-1} D_{22}f(p_i,0)(v_{i+k},v_{i+k'})+\sum_{j\in\mathbb{Z}} a_{-jN}(\sum_{k\le j-1} D_{22}f(p_i,0)(v_{i+k+1},v_i)$$
$$+\sum_{k'\le j-1} D_{22}f(p_i,0)(v_i,v_{i+k'+1}))+D_{22}f(p_i,0)(v_i,v_i).$$

Moreover for $l=0,\dots,N-1$

$$2Tp_{l+iN} \overset{(2)}{=} \sum_{k,k'\in\mathbb{Z}} a_{l-kN}a_{l-k'N}D_{22}f(p_i,0)(v_i+\cdots+v_{i+k-1},v_i+\cdots+v_{i+k'-1}),$$

and for $l = N - 1$

$$
\begin{aligned}
2Tp_{l+1+iN} &\overset{(2)}{=} \sum_{k,k' \in \mathbb{Z}} a_{-kN} a_{-k'N} D_{22} f(p_i, 0)(v_{i+1} + \cdots + v_{i+k}, v_{i+1} + \cdots + v_{i+k'}) \\
&+ \sum_{k' \in \mathbb{Z}} a_{-k'N} D_{21} f(p_i, 0)(D_2 f(p_i, 0) v_i, v_{i+1} + \cdots + v_{i+k'}) \\
&+ \sum_{k \in \mathbb{Z}} a_{-kN} D_{21} f(p_i, 0)(v_{i+1} + \cdots + v_{i+k}, D_2 f(p_i, 0) v_i) \\
&+ D_{22} f(p_i, 0)(v_i, v_i) \\
&= \sum_{k,k' \in \mathbb{Z}} a_{N-kN} a_{N-k'N} D_{22} f(p_i, 0)(v_i + \cdots + v_{i+k-1}, v_i + \cdots + v_{i+k'-1}) \\
&+ \sum_{k' \in \mathbb{Z}} a_{-k'N} D_{22} f(p_i, 0)(v_i, v_{i+1} + \cdots + v_{i+k'}) \\
&+ \sum_{k \in \mathbb{Z}} a_{-kN} D_{22} f(p_i, 0)(v_{i+1} + \cdots + v_{i+k}, v_i) \\
&+ D_{22} f(p_i, 0)(v_i, v_i).
\end{aligned}
$$

Now denoting $c_l^{k,k'} = (\sum_{k \leq j-1} a_{l+1-jN} \sum_{k' \leq j-1} a_{l+1-jN} - \sum_{k \leq j-1} a_{l-jN} \sum_{k' \leq j-1} a_{l-jN}) - \sum_{k,k' \leq j-1} (a_{l+1-jN} - a_{l-jN})$ and writing the second order Taylor expansion of $\Delta(Tp - Sp)_{l+iN}$ as

$$\Delta(Sp - Tp)_{l+iN} = P(v) + o(\|v\|^3)$$

we arrive at

$$P(v) = \sum_{k,k' \in \mathbb{Z}} A_{k,k'}(v_{i+k}, v_{i+k'})$$

where

$$A_{k,k'} = c_l^{k,k'} D_{22} f(p_i, 0).$$

Note that

$$P(v) = \sum_{k,k' \in \mathbb{Z}} A_{k,k'}(v_0, v_0) + \sum_{k,k' \in \mathbb{Z}} (\sum_{l=0}^{k-1} A_{k,k'}(\Delta v_l, v_j) + \sum_{l=0}^{k'-1} A_{k,k'}(v_0, \Delta v_l)).$$

In view of Lemma 1 we further have $\sum_{k,k'\in\mathbb{Z}} A_{k,k'} = \sum_{k,k'\in\mathbb{Z}} c_l^{k,k'} D_{22} f(p_i, 0) = 0$. Hence we can write

$$\Delta(Sp - Tp)_{l+iN} = \sum_{k,k'\in\mathbb{Z}} (\sum_{l=0}^{k-1} A_{k,k'}(\Delta v_l, v_j) + \sum_{l=0}^{k'-1} A_{k,k'}(v_0, \Delta v_l)) + o(\|v\|^3).$$

Applying the triangle inequality to the above equation and using lemma 2 we immediately arrive at the desired estimate. □

3.2.Main results

In this section we present our main results as follows. [1]If the derived schemes of the linear subdivision scheme S up to order 3 exist and their norm satisfy certain inequalities, then the analogues nonlinear scheme is C^2.

Theorem 4. *(i) If S_2 exists and*

$$\mu_0^2 < \frac{\mu_1}{N}, \tag{17}$$

then T is C^1 equivalent to S.
(ii) Suppose that S_2 and S_3 exist and beside (17) *the inequality*

$$\mu_0 \mu_1 < \frac{\mu_2}{N} \tag{18}$$

holds. Then T is C^2 equivalent to S.

Proof. Note that (17) and (18) imply

$$\mu_0^3 < \frac{\mu_2}{N^2}.$$

In view of the proximity conditions (14) and (15) we can apply theorem 6 of [6] to deduce the results. □

[1]Recall that by affine invariance, first derived scheme of the linear subdivision scheme S always exist.

We denote that the inequlaities (17) and (18) are particularly fulfilled for $\mu_i = 1/N$ which is true for B-Splines.

4. Examples

Data from meaurments of poses of rigid body live in the Euclidean motion group. An example of log-exponential subdivision in this case has been represented in [9]. For applications of geodesic subdivision in surfaces, e.g. hyperbolic plane, we refer to [7]. Here we present some further examples.

Example 5. It is well-known (see for instance [1] and [3]) that the space of positive definite symmetric matrices is a Riemannian symmetric space. Moreover, it is also a Hadamard manifold with exponential map (globally) given by $Exp_p(v) = p^{1/2}exp(p^{-1/2}vp^{-1/2})p^{1/2}$. Choosing $f = Exp$ and applying 2 rounds of qubic B-spline geodesic subdivision we get the following figure. Note that each ellipsoid represents a positive definite symmetric matrix.

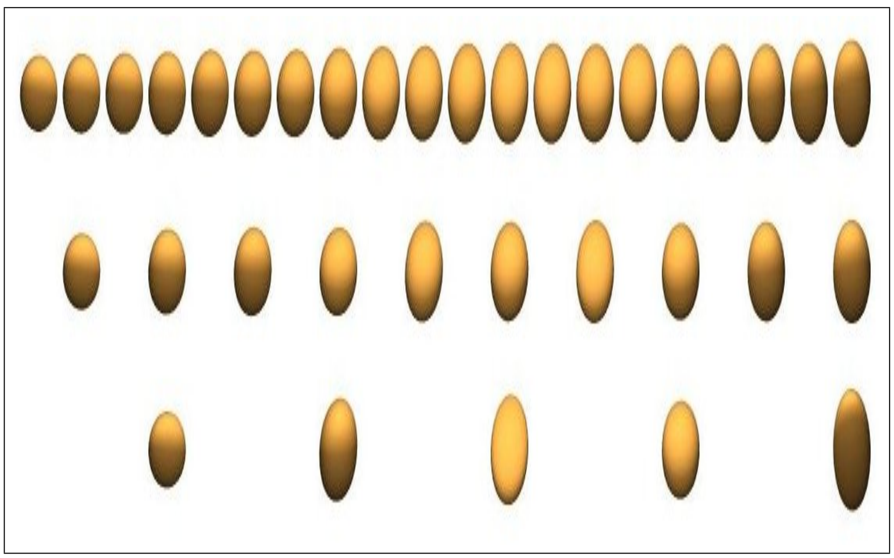

Figure 1: Subdivision in the space of positive matrices

Example 6. In the following we consider as a prominent example of a Hilbert manifold the loop space of $\mathbb{R}^2$ and apply qubic B-spline geodesic subdivision to the polygon p consisting of loops p_1, p_2 and p_3 through a fixed pint. Figure 2 shows the result after 1 and 2 round of subdivision.

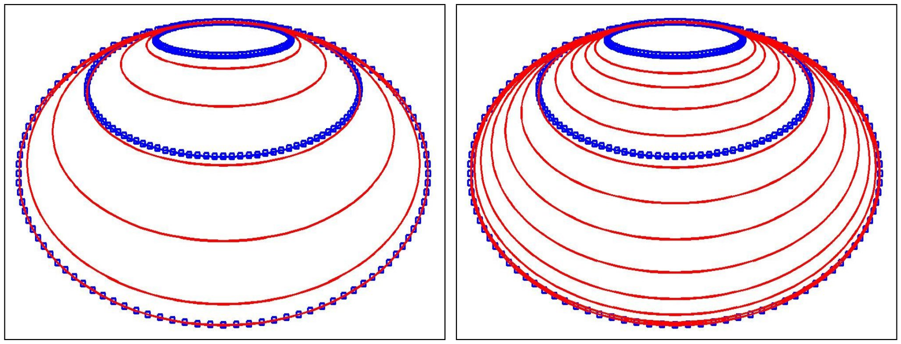

Figure 2: Subdivision in the loop space

5.Conclusions and remarks

We have established the notion of analogues of a linear subdivision operating on manifold-valued data. Furthermore we have investigated smoothness properties of such subdivision schemes. Particulrly, we have shown that if the linear scheme enjoys C^2 smoothness and its derived schemes up to order 3 satisfy certain boundedness inequalities, then the nonlinear analogues one is also C^2.

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