SOME INTEGRAL OPERATORS AND THEIR UNIVALENCE

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Abstract. In this work we obtain the conditions of univalence for the analicity and univalence in the unit disc $\mathcal{U} = \{z \in \mathbb{C}, |z| < 1\}$ of certain integral operators.

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1.Introduction

Let $\mathcal{A}$ be the class of the of the functions f which are analytic in the unit disc $\mathcal{U}$ and $f(0) = f'(0) - 1 = 0$. We denote by $\mathcal{S}$ the class of the functions $f \in \mathcal{A}$ which are analytic in $\mathcal{U}$.

Ozaki and Nunokawa [2] investigated the univalence of the functions $f \in \mathcal{A}$.

Theorem A. *Let $f \in \mathcal{A}$ satisfy the condition:*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1 \tag{1}$$

for all $z \in \mathcal{U}$, then, f is univalent in $\mathcal{U}$.

2.Preliminary Results

We need the following theorem and lemma for proving our main results.

Theorem B.[**3**]*Let* α *be a complex number,* $\mathrm{Re}\alpha > 0$ *and* $f \in \mathcal{A}$*. If*

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{2}$$

for all $z \in \mathcal{U}$*, then the function*

$$F_\alpha(z) = \left[\alpha\int_0^z u^{\alpha-1}f'(u)du\right]^{\frac{1}{\alpha}} \tag{3}$$

is in the class $\mathcal{S}$*.*

The Schwarz Lemma. [**1**]*Let the analytic function* f *be regular in the unit disk and let* $f(0) = 0$*. If* $|f(z)| \leq 1$*, then*

$$|f(z)| \leq |z| \tag{4}$$

for all $z \in \mathcal{U}$*, where the equality can hold only if* $|f(z)| \equiv Kz$ *and* $K = 1$*.*

3.Main results

Theorem 1.*Let* $g \in \mathcal{A}$ *verifies (1) and* $a + bi$ *be a complex number,* a, b *satisfies the conditions*

$$a \in \left(0, \sqrt{3}\right] \tag{5}$$

$$a^4 + a^2b^2 - 9 \geq 0. \tag{6}$$

If

$$|g(z)| \leq 1 \tag{7}$$

for all $z \in \mathcal{U}$*, then the function*

$$F(z) = \left[(a+bi)\int_0^z u^{a+bi-1}\left(\frac{g(u)}{u}\right)^{\frac{1}{a+bi}} du\right] \tag{8}$$

is in the class $\mathcal{S}$*.*

Proof. Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\frac{1}{a+bi}} du. \tag{9}$$

The function f is regular in $\mathcal{U}$.

From (9) we have

$$f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{a+bi}}$$

$$f''(z) = \frac{1}{a+bi}\left(\frac{g(z)}{z}\right)^{\frac{1}{a+bi}-1} \frac{zg'(z)-g(z)}{z^2}$$

and

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| = \frac{1-|z|^{2a}}{a}\frac{1}{\sqrt{a^2+b^2}}\left|\frac{zg'(z)}{g(z)}-1\right| \tag{10}$$

for all $z \in \mathcal{U}$.

From (10) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}}\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right) \tag{11}$$

for all $z \in \mathcal{U}$, and hence, we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}}\left(\left|\frac{z^2g'(z)}{g^2(z)}\right|\left|\frac{g(z)}{z}\right|+1\right) \tag{12}$$

for all $z \in \mathcal{U}$.

By the Schwarz-Lemma and using (12) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}}\left(\left|\frac{z^2g'(z)}{g^2(z)}-1\right|+2\right) \tag{13}$$

for all $z \in \mathcal{U}$.

From (13) and because g verifies the condition (1) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{3\left(1-|z|^{2a}\right)}{a\sqrt{a^2+b^2}} \le \frac{3}{a\sqrt{a^2+b^2}} \tag{14}$$

for all $z \in \mathcal{U}$.

From (5) and (6) we have

$$\frac{3}{a\sqrt{a^2+b^2}} \leq 1. \tag{15}$$

Using (15) and (14) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1. \tag{16}$$

for all $z \in \mathcal{U}$.

From (9) we obtain $f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{a+bi}}$ and by Theorem B it results that the function F is in the class $\mathcal{S}$.

Theorem 2. *Let $g \in \mathcal{A}$ verifies (1), $a+bi$ be a complex number, a, b satisfies the conditions*

$$a \in \left[\frac{3}{4}, \frac{3}{2}\right], \quad b \in \left[0, \frac{1}{2\sqrt{2}}\right] \tag{17}$$

and

$$8a^2 + 9b^2 - 18a + 9 \leq 0. \tag{18}$$

If

$$|g(z)| \leq 1 \tag{19}$$

for all $z \in \mathcal{U}$, then the function

$$G(z) = \left\{(a+bi)\int_0^z [g(u)]^{a+bi-1}\, du\right\}^{\frac{1}{a+bi}} \tag{20}$$

is in the class $\mathcal{S}$.

Proof. From (20) we have

$$G(z) = \left\{(a+bi)\int_0^z u^{a+bi-1}\left(\frac{g(u)}{u}\right)^{a+bi-1} du\right\}^{\frac{1}{a+bi}}. \tag{21}$$

Let us consider the function

$$p(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{a+bi-1} du. \tag{22}$$

The function p is regular in $\mathcal{U}$.

From (22) we get $p'(z) = \left(\frac{g(z)}{z}\right)^{a+bi-1}$, and

$p''(z) = (a+bi-1)\left(\frac{g(z)}{z}\right)^{a+bi-2} \frac{zg'(z)-g(z)}{z^2}$ and

$$\frac{1-|z|^{2a}}{a}\left|\frac{zp''(z)}{p'(z)}\right| \leq \frac{1-|z|^{2a}}{a}|a+bi-1|\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right) \tag{23}$$

for all $z \in \mathcal{U}$, and hence, we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zp''(z)}{p'(z)}\right| \leq \sqrt{(a-1)^2+b^2}\,\frac{1-|z|^{2a}}{a}\left(\left|\frac{z^2g'(z)}{g^2(z)}\right|\frac{|g(z)|}{|z|}+1\right) \tag{24}$$

By the Schwarz-Lemma and using (24) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zp''(z)}{p'(z)}\right| \leq \sqrt{(a-1)^2+b^2}\,\frac{1-|z|^{2a}}{a}\left(\left|\frac{z^2g'(z)}{g^2(z)}-1\right|+2\right) \tag{25}$$

From (25) and since g satisfies the condition (1) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zp''(z)}{p'(z)}\right| \leq 3\sqrt{(a-1)^2+b^2}\,\frac{1-|z|^{2a}}{a} \leq \frac{3\sqrt{(a-1)^2+b^2}}{a} \tag{26}$$

for all $z \in \mathcal{U}$.

From (17) and (18) we get

$$\frac{3\sqrt{(a-1)^2+b^2}}{a} \leq 1 \tag{27}$$

and by (26) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zp''(z)}{p'(z)}\right| \leq 1 \tag{28}$$

for all $z \in \mathcal{U}$.

From (22) we have $p'(z) = \left(\frac{g(z)}{z}\right)^{a+bi-1}$ and by Theorem B it results that the function G is in the class $\mathcal{S}$.

References

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