A CONVEXITY PROPERTY FOR AN INTEGRAL OPERATOR ON THE CLASS $\mathcal{UST}(K,\gamma)$

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Abstract. An integral operator $F_n(z)$ for analytic functions $f_i(z)$ in the open unit disk $\mathbb{U}$ is introduced. The object of the present paper is to discuss a convexity property for the integral operator $F_n(z)$ involving k-uniformly starlike functions of order γ.

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1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}^*(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < 1)$. A function $f(z) \in \mathcal{S}^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$. Also let $\mathcal{K}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < 1)$. A function $f(z)$ in $\mathcal{K}(\alpha)$ is said to be convex of order α in $\mathbb{U}$. Further let $\mathcal{UST}(k,\gamma)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma \qquad (z \in \mathbb{U})$$

for some k ($k \geqq 0$) and γ ($0 \leqq \gamma < 1$). A function $f(z) \in \mathcal{UST}(k, \gamma)$ is said to be k-uniformly starlike of order γ. This class was introduced by Aghalary and Azadi [1] (also see [3]).

For $f_i(z) \in \mathcal{A}$ and $\alpha_i > 0$, we define the integral operator $F_n(z)$ given by

$$F_n(z) = \int_0^z \left(\prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} \right) dt.$$

This integral operator $F_n(z)$ was first defined by Breaz and Breaz [2]. From the definition for the integral operator, it is easy to see that $F_n(z) \in \mathcal{A}$.

2. Main Result

Our main result for the integral operator $F_n(z)$ is contained in

Theorem 1. *Let $f_i(z) \in \mathcal{UST}(k_i, \gamma_i)$ for each $i = 1, 2, 3, \cdots, n$. If $\alpha_i > 0$ ($i = 1, 2, 3, \cdots, n$) and*

$$0 < \sum_{i=1}^{n} (1 - \gamma_i)\alpha_i \leqq 1,$$

then $F_n(z) \in \mathcal{K}(\gamma)$, where

$$\gamma = 1 - \sum_{i=1}^{n} (1 - \gamma_i)\alpha_i.$$

Proof. Since

$$F_n'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z} \right)^{\alpha_i}$$

and

$$F_n''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z} \right)^{\alpha_i - 1} \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)} \right) \prod_{j=1 j \neq i}^{n} \left(\frac{f_j(z)}{z} \right)^{\alpha_j},$$

we obtain

$$1 + \frac{F_n''(z)}{F_n'(z)} = 1 + \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right).$$

Thus we see, for $f_i(z) \in \mathcal{UST}(k_i, \gamma_i)$,

$$\mathrm{Re} \left(1 + \frac{F_n''(z)}{F_n'(z)} \right) = 1 + \sum_{i=1}^{n} \alpha_i \mathrm{Re} \left(\frac{z f_i'(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i$$

$$> 1 + \sum_{i=1}^{n} \alpha_i \left(k_i \left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| + \gamma_i \right) - \sum_{i=1}^{n} \alpha_i$$

$$\geqq 1 - \sum_{i=1}^{n} (1 - \gamma_i) \alpha_i .$$

This completes the proof of our theorem.

Corollary 2. *Let $f(z) \in \mathcal{UST}(k, \gamma)$. If $\alpha > 0$ and*

$$0 < \alpha \leqq \frac{1}{1 - \gamma},$$

then

$$F_1(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} dt \in \mathcal{K}(\delta),$$

where $\delta = 1 - (1 - \gamma)\alpha$.

Remark 3. If we take $\alpha = 1$ in the corollary, then the Alexander operator defined by

$$F(z) = \int_0^z \frac{f(t)}{t} \, dt$$

is convex of order γ. Therefore, letting $k = 0$, we see that $f(z) \in \mathcal{S}^*(\gamma)$ implies that

$$F(z) = \int_0^z \frac{f(t)}{t} \, dt \in \mathcal{K}(\gamma).$$

References

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