ON SOME RESULTS FOR $\lambda-$SPIRALLIKE FUNCTIONS OF COMPLEX ORDER OF HIGHER-ORDER DERIVATIVES OF MULTIVALENT FUNCTIONS

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Abstract. In this paper we establish results of various kinds concerning $\lambda-$ spirallike functions of complex order in the unit disc $\mathcal{U} = \{z : z \in \mathcal{C}, |z| < 1\}$ by using the method of differential subordination. Our results provide extensions and generalizations of many known and new results.

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1. Introduction

For a fixed $p \in \mathcal{N} := \{1, 2, 3, ...\}$, let $\mathcal{A}_{n,p}$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \qquad (n \in \mathcal{N}), \tag{1.1}$$

which are p - valent in the open unit disc $\mathcal{U} = \{z : z \in \mathcal{C}, |z| < 1\}$. Upon differentiating both sides of (1.1) q - times with respect to z, the following differential operator is obtained :

$$f^{(q)}(z) = \alpha(p;q)\ z^{p-q} + \sum_{k=n+p}^{\infty} \alpha(k;q)\ a_k z^{k-q}, \tag{1.2}$$

where

$$\alpha(p;q) = \frac{p!}{(p-q)!} \quad (p \geq q;\ p \in \mathcal{N};\ q \in \mathcal{N} \cup \{0\}). \tag{1.3}$$

Several researchers have investigated higher-order derivatives of multivalent functions, (see, e.g.[8,12]).

A function $f(z) \in \mathcal{A}_{n,p}$ is said to be $\lambda-$ spirallike of complex order in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left\{\frac{1}{b\cos\lambda}\left[e^{i\lambda}\left\{\frac{zf^{(q+1)}(z)}{(p-q)f^{(q)}(z)}\right\}-(1-b)\cos\lambda-i\sin\lambda\right]\right\}>\rho \tag{1.4}$$

$$(p \geq q;\ p \in \mathcal{N};\ q \in \mathcal{N}\cup\{0\};\ 0 \leq \rho < 1),$$

for some real λ, $|\lambda| < \frac{\pi}{2}$, $b \neq 0$, complex.

We denote this class by $\mathcal{S}_n^{\lambda,q}(\rho; b)$.

Now we mention below some known subclasses of our class $\mathcal{S}_n^{\lambda,q}(\rho; b)$:

(i) For $q = 0,\ p = 1,\ \rho = 0$ and $n = 1$, our class is reduced into the class $\mathcal{S}^{\lambda}(b)$, introduced and studied by Al-oboudi and Haidan [1].

(ii) For $q = 0,\ p = 1,\ \rho = 0$ and $b = 1-\delta$ $(0 \leq \delta < 1)$, our class is reduced into the class $\mathcal{S}_n^{\lambda}(\delta)$, introduced by Obradovic and Owa [9]. Further this class contains the classes due to Libera [4] (for n=1) and Spacek [13] (for n=1, $\delta = 0$) as special cases.

(iii) For $q = 0,\ p = 1,\ \lambda = 0$ and $n = 1$, our class is reduced into starlike of complex order b and type ρ denoted by $\mathcal{S}_{\rho}^{*}(b)$, which contain another class by Nasr and Aouf [7], for $\rho = 0$.

Let f and g be analytic in the unit disc $\mathcal{U}$. The function f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

The general theory of differential subordinations was introduced by Miller and Mocanu [5]. Some classes of the first-order differential subordinations were considered by the same authors in [5]. Namely let $\psi : \mathcal{C}^2 \to \mathcal{C}$ ($\mathcal{C}$ is the complex plane) be analytic in a domain D, let h be univalent in $\mathcal{U}$, and let $p(z)$ be analytic in $\mathcal{U}$ with $(p(z), zp^{'}(z)) \in D$ when $z \in \mathcal{U}$, then $p(z)$ is said to satisfy the first order differential subordination if

$$\psi\left((p(z), zp'(z)\right) \prec h(z) \tag{1.5}$$

The univalent function q is said to be a dominant of the differential subordination (1.5) if $p \prec q$ for all p satisfying (1.5). If $\tilde{q}$ is a dominant of (1.5) and $\tilde{q} \prec q$ for all dominants q of (1.5), then q is said to be best dominant of (1.5).

2. Results and Consequences.

We shall require the following results due to Miller and Mocanu in order to prove our main results of the next section:

Lemma 1 [6]. *Let q be univalent in the unit disk $\mathcal{U}$, and let θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$.*
Set $Q(z) = zq'(z)\phi(q(z)),\ h(z) = \theta(q(z)) + Q(z)$ and suppose that

(i) Q is starlike (univalent) in $\mathcal{U}$ with $Q(0) = 0$ and $Q^{'}(0) \neq 0$.
(ii) $Re\{zh'(z)/Q(z)\} > 0$ for $z \in \mathcal{U}$.
If p is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

Lemma 2 [5]. *Let $\phi(u, v)$ be complex valued function, $\phi : D \to \mathcal{C}$, $D \subset \mathcal{C} \times \mathcal{C}$ and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies the following conditions:*
(i) $\phi(u, v)$ is continuous in D;
(ii) $(1, 0) \in D$ and Re $\{\phi(1, 0)\} > 0$;
(iii) Re $\{\phi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -n(1 + u_2^2)/2$.
Suppose that $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + ...$, be regular in the unit disk $\mathcal{U}$ such that
$(p(z), zp^{'}(z)) \in D$ for all $z \in \mathcal{U}$. If

$$Re\left\{\phi(p(z), zp^{'}(z))\right\} > 0 \quad (z \in \mathcal{U}),$$

then

$$Re\{p(z)\} > 0 \quad (z \in \mathcal{U}).$$

Lemma 3 [11]. *The function $(1 - z)^{\gamma} \equiv e^{\gamma \log(1-z)}, \gamma \neq 0$ is univalent in $\mathcal{U}$ if and only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.*

3. Main Results and Consequences.

Theorem 1. *Let $f \in \mathcal{S}_n^{\lambda,q}(\rho; b)$, $(|\lambda| < \frac{\pi}{2}$, $b \neq 0$, complex, $0 \leq \rho < 1$, $p > q)$, then*

$$\left(\frac{f^{(q)}(z)}{\alpha(p; q)\ z^{p-q}}\right)^{\gamma} \prec \frac{1}{(1 - z)^{2\gamma(p-q)(1-\rho)b\cos\lambda e^{-i\lambda}}} \tag{3.1}$$

where $\gamma \neq 0$ is complex and either $|2\gamma(p - q)(1 - \rho)b \cos\lambda e^{-i\lambda} + 1| \leq 1$ or $|2\gamma(p - q)(1 - \rho)b \cos\lambda e^{-i\lambda} - 1| \leq 1$ and this is the best dominant.

Proof. Let $q(z) = (1 - z)^{-2\gamma(p-q)(1-\rho)b\cos\lambda e^{-i\lambda}}$, $\phi(w) = \left(\gamma b(p - q)\cos\lambda e^{-i\lambda}\right)^{-1} w^{-1}$ and $\theta(w) = 1$ in Lemma 1. Then it is easy to verify the conditions (i) and (ii) of the Lemma 1. Namely q is univalent in $\mathcal{U}$ by Lemma 3, while

$$h(z) = \theta(q(z)) + zq'(z)\phi(q(z)) = \frac{1 + (1 - 2\rho)z}{1 - z}.$$

Consequently, for $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + ...$, analytic in $\mathcal{U}$ with $p(z) \neq 0$ for $0 < |z| < 1$, from (2.1) we get

$$1 + \frac{e^{i\lambda}}{\gamma b(p-q)\cos\lambda} \frac{zp'(z)}{p(z)} \prec \frac{1 + (1-2\rho)z}{1-z} \tag{3.2}$$

$$\Rightarrow p(z) \prec q(z)$$

Now, if in (3.2) we choose $p(z) = \left(\frac{f^{(q)}(z)}{\alpha(p;q)\ z^{p-q}}\right)^{\gamma}$, then we have

$$\left(\frac{f^{(q)}(z)}{\alpha(p;q)\ z^{p-q}}\right)^{\gamma} \prec \frac{1}{(1-z)^{2\gamma(p-q)(1-\rho)b\cos\lambda e^{-i\lambda}}}$$

which evidently, completes the proof of Theorem 1.

If we put $\gamma = \frac{-e^{i\lambda}}{2(p-q)(1-\rho)b\cos\lambda}$ in Theorem 1, we get

Corollary 2. *Let* $f(z) \in \mathcal{S}_n^{\lambda,q}(\rho; b)$, $(|\lambda| < \frac{\pi}{2}$, $b \neq 0$, *complex*, $0 \leq \rho < 1$, $p > q)$, *then*

$$\left(\frac{\alpha(p;q)\ z^{p-q}}{f^{(q)}(z)}\right)^{\frac{e^{i\lambda}}{2(p-q)(1-\rho)b\cos\lambda}} \prec (1-z) \tag{3.3}$$

and this is the best dominant.

From (3.3), we have the following inequality for $f(z) \in \mathcal{S}_n^{\lambda,q}(\rho; b)$

$$\left|\left(\frac{\alpha(p;q)\ z^{p-q}}{f^{(q)}(z)}\right)^{\frac{e^{i\lambda}}{2(p-q)(1-\rho)b\cos\lambda}} - 1\right| \leq |z| \quad (z \in \mathcal{U}) \tag{3.4}$$

Remark. (i) If we put $q = 0$, $p = 1$ and $\rho = 0$ in Theorem 1, we get a recent result by Aouf, Oboudi and Haidan [2] for the class $\mathcal{S}^{\lambda}(b)$, which contains the results obtained earlier by Obradovic, Aouf and Owa [10] for the classes $\mathcal{S}(b)$, $\mathcal{S}^{\lambda}$ and $\mathcal{S}^{\lambda}(\delta)$ respectively.

(ii) Putting $q = 0$, $p = 1$ and $\rho = 0$ in Corollary 2, we get a result obtained by Aouf, Al- Oboudi and Haidan [2]. Also $\lambda = 0$ in (3.4), gives the result obtained earlier by Obradovic, Aouf and Owa [10].

Theorem 3. *Let the function* $f(z)$ *defined by (1.1) be in the clas* $\mathcal{S}_n^{\lambda,q}(\rho; b)$ *and let*

$$0 < \beta \leq \frac{n}{2(p-q)(1-\rho)b\cos\lambda} \tag{3.5}$$

Then we have

$$\text{Re}\left\{\left(\frac{f^{(q)}(z)}{\alpha(p;q)\ z^{p-q}}\right)^{\beta e^{i\lambda}}\right\} > \frac{n}{2\beta(p-q)(1-\rho)b\cos\lambda+n} \quad (z\in\mathcal{U}),$$

where $b\neq 0$ *real*, $|\lambda|<\frac{\pi}{2},\ 0\leq\rho<1,\ p>q,\ p\in\mathcal{N},$ *and* $q\in\mathcal{N}\cup\{0\}$.

Proof. If we put

$$B=\frac{n}{2\beta(p-q)(1-\rho)b\cos\lambda+n} \tag{3.6}$$

and

$$\left(\frac{f^{(q)}(z)}{\alpha(p;q)\ z^{p-q}}\right)^{\beta e^{i\lambda}}=(1-B)\,p(z)+B \qquad (3.7),$$

where β satisfied (3.5) then $p(z)=1+p_nz^n+p_{n+1}z^{n+1}+...$, is regular in the unit disk $\mathcal{U}$.

By a simple computation, we observe from (3.7) that

$$e^{i\lambda}\left[\frac{zf^{(q+1)}(z)}{(p-q)f^{(q)}(z)}-1\right]=\frac{(1-B)zp'(z)}{\beta(p-q)\left[(1-B)p(z)+B\right]} \tag{3.8}$$

and from that

$$\frac{1}{b\cos\lambda}\left[e^{i\lambda}\left\{\frac{zf^{(q+1)}(z)}{(p-q)f^{(q)}(z)}-1\right\}+b\cos\lambda\right]-\rho=1-\rho+\frac{(1-B)zp'(z)}{\beta(p-q)b\cos\lambda\left[(1-B)p(z)+B\right]} \tag{3.9}$$

Since $f(z)\in\mathcal{S}_n^{\lambda,q}(\rho;b)$, therefore from (3.9), we get

$$Re\left[1-\rho+\frac{(1-B)zp'(z)}{\beta(p-q)b\cos\lambda\left\{(1-B)p(z)+B\right\}}\right]>0,\quad (z\in\mathcal{U}) \tag{3.10}$$

Let us consider the function $\phi(u,v)$ defined by

$$\phi(u,v)=1-\rho+\frac{(1-B)v}{\beta(p-q)b\cos\lambda\left[(1-B)u+B\right]}\ ,$$

then $\phi(u,v)$ is continuous in $D=\mathcal{C}-\left\{\frac{-B}{1-B}\right\}\times\mathcal{C}$.

Also, $(1,0)\in D$ and $Re\left\{\phi(1,0)\right\}=1-\rho>0$.

Furthermore, for all $(iu_2,v_1)\in D$ such that $v_1\leq\frac{-n(1+u_2^2)}{2}$, we have

$$\text{Re}\left\{\phi(iu_2,v_1)\right\}=1-\rho+\text{Re}\left[\frac{(1-B)v_1}{\beta(p-q)b\cos\lambda\left\{(1-B)iu_2+B\right\}}\right]$$

$$= 1 - \rho + \frac{B(1-B)v_1}{\beta(p-q)b\cos\lambda\left\{(1-B)^2u_2^2 + B^2\right\}}$$

$$\leq 1 - \rho - \frac{nB(1-B)(1+u_2^2)}{2\beta(p-q)b\cos\lambda\left\{(1-B)^2u_2^2 + B^2\right\}}$$

$$= \frac{(1-B)\left[2\beta(p-q)(1-\rho)b\cos\lambda - n\right]u_2^2}{2\beta(p-q)b\cos\lambda\left\{(1-B)^2u_2^2 + B^2\right\}} \leq 0$$

because $0 < B < 1$ and $(2\beta(p-q)(1-\rho)b\cos\lambda - n) \leq 0$.

Therefore, the function $\phi(u,v)$ satisfies the condition of Lemma 2. This proves that
$Re\{p(z)\} > 0$ for $z \in \mathcal{U}$, that is, from (3.7)

$$\text{Re}\left\{\left(\frac{f^{(q)}(z)}{\alpha(p;q)\; z^{p-q}}\right)^{\beta e^{i\lambda}}\right\} > B \quad (z \in \mathcal{U}),$$

which is equivalent to the statement of Theorem 3.

Taking $\rho = 0$ and $\beta = \frac{n}{2(p-q)b\cos\lambda}$ in Theorem 3, we have

Corollary 4. *Let the function $f(z)$ defined by (1.1) be in the $\mathcal{S}_n^{\lambda,q}(0;b)$. Then*

$$\text{Re}\left\{\left(\frac{f^{(q)}(z)}{\alpha(p;q)\; z^{p-q}}\right)^{\frac{ne^{i\lambda}}{2(p-q)b\cos\lambda}}\right\} > \frac{1}{2} \quad (z \in \mathcal{U}).$$

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References

[1] F. M. Al-Oboudi and M. M. Haidan, *Spirallike functions of complex order*, J. Natural Geom., **19** (2000), 53-72.

[2] M. K. Aouf, F. M. Al-Oboudi and M. M. Haidan, *On some results for $\lambda-$spirallike and $\lambda-$ Robertson functions of complex order*, Publ. Inst. Math., Nouv. ser., **75** (2005), 93-98.

[3] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Moscow, (1952), English Translation, Proc. Amer. Math. Soc., Providence, R.I., (1969).

[4] R. J. Libera, *Univalent $\lambda-$ spiral functions*, Canad. J. Math., **19** (1967), 449-456.

[5] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. , **65** (1978), 289-305.

[6] S. S. Miller and P. T. Mocanu, *On some classes of first-order differential subordinations*, Michigan Math. J., **32** (1985), 185-195.

[7] M. A. Nasr and M. K. Aouf, *Starlike functions of complex order*, J. Natural Sci. Math., **25** (1985), 1-12.

[8] M. Nunokawa, *On the multivalent functions*, Indian J. Pure Appl. Math., **20** (1989), 577-582.

[9] M. Obradovic and S. Owa, *On some results for $\lambda-$ spiral functions of order α*, Int. J. Math. Math. Sci., **9** (1986), 439-446.

[10] M. Obradovic, M. K. Aouf and S. Owa, *On some results for starlike functions of complex order*, Publ. Inst. Math., Nouv. Ser., **46** , (1989), 79-85.

[11] M. S. Robertson, *Certain classes of integral operators*, J. Natur. Geom., **16**, (1999).

[12] H. Silverman, *Higher order derivatives*, Chi. J. of Math., **23**, (1995), 189-191.

[13] L. Spacek, *Contribution a la theori des function univalents*, Casopis Pest. Math., (1932), 12-19.

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