THE INTEGRAL OPERATOR ON THE $SH(\beta)$ CLASS

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Abstract. In this paper we present a convexity condition for a integral operator F defined in formula (2) on the class $SH(\beta)$.

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1. 1.Introduction

Let $U = \{z \in C, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$, the class of the olomorphic functions in U. Consider $A = \{f \in H(U), f(z) = z + a_2z^2 + a_3z^3 + ..., z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Denote with K the class of the olomorphic functions in U with $f(0) = f'(0) - 1 = 0$, where is convex functions in U , defined by

$$K = \left\{ f \in H(U); f(0) = f'(0) - 1 = 0, \mathbf{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0, z \in U \right\}.$$

In the paper (3) J. Stankiewicz and A. Wisniowska introduced the class of univalent functions $SH(\beta)$, define by the next inequality:

$$\left| \frac{zf'(z)}{f(z)} - 2\beta\left(\sqrt{2} - 1\right) \right| < \mathbf{Re}\left\{ \sqrt{2}\frac{zf'(z)}{f(z)} \right\} + 2\beta\left(\sqrt{2} - 1\right), \qquad (1)$$

for some $\beta > 0$ and for all $z \in U$.

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We consider the integral operator

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{2}$$

and we study your properties.

Remark. We observe that for $n = 1$ and $\alpha_1 = 1$ we obtain the integral operator of Alexander.

2.Main results

Theorem 1. *Let $\alpha_i, i \in \{1, ..., n\}$ the real numbers with the properties $\alpha_i > 0$ for $i \in \{1, ..., n\}$ and*

$$\sum_{i=1}^{n} \alpha_i \leq \frac{\sqrt{2}}{2\beta\left(\sqrt{2}-1\right)+\sqrt{2}}. \tag{3}$$

We suppose that the functions $f_i \in SH(\beta)$ for $i = \{1, ..., n\}$ and $\beta > 0$. In this conditions the integral operator defined in (2) is convex.

Proof. We calculate for F the derivatives of the first and second order.
From (2) we obtain:

$$F'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)}\right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}$$

$$\frac{F''(z)}{F'(z)} = \alpha_1 \left(\frac{z f_1'(z) - f_1(z)}{z f_1(z)}\right) + \ldots + \alpha_n \left(\frac{z f_n'(z) - f_n(z)}{z f_n(z)}\right).$$

$$\frac{F''(z)}{F'(z)} = \alpha_1 \left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \ldots + \alpha_n \left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right). \tag{4}$$

Multiply the relation (4) with z we obtain:

$$\frac{zF''(z)}{F'(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i. \tag{5}$$

The relation (5) is equivalent with

$$\frac{zF''(z)}{F'(z)}+1=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i+1. \tag{6}$$

We multiply the relation (6) with $\sqrt{2}$ and obtain:

$$\sqrt{2}\left(\frac{zF''(z)}{F'(z)}+1\right)=\sum_{i=1}^{n}\sqrt{2}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}. \tag{7}$$

The equality (7) is equivalent with:

$$\begin{array}{c}\sqrt{2}\left(\frac{zF''(z)}{F'(z)}+1\right)=\alpha_1\sqrt{2}\frac{zf_1'(z)}{f_1(z)}+2\alpha_1\beta\left(\sqrt{2}-1\right)+...\\+\alpha_n\sqrt{2}\frac{zf_n'(z)}{f_n(z)}+2\alpha_n\beta\left(\sqrt{2}-1\right)-\\-\sum_{i=1}^{n}2\alpha_i\beta\left(\sqrt{2}-1\right)-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}.\end{array}$$

We calculate the real part from both terms of the above equality and obtain:

$$\begin{array}{c}\sqrt{2}\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)=\alpha_1\left(\mathbf{Re}\left\{\sqrt{2}\frac{zf_1'(z)}{f_1(z)}\right\}+2\beta\left(\sqrt{2}-1\right)\right)+...\\+\alpha_n\left(\mathbf{Re}\left\{\sqrt{2}\frac{zf_n'(z)}{f_n(z)}\right\}+2\beta\left(\sqrt{2}-1\right)\right)-\\-\sum_{i=1}^{n}2\alpha_i\beta\left(\sqrt{2}-1\right)-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}.\end{array}$$

Because $f_i\in SH(\beta)$ for $i=\{1,...,n\}$ we apply in the above relation the inequality (1) and obtain:

$$\begin{array}{c}\sqrt{2}\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)>\alpha_1\left|\frac{zf_1'(z)}{f_1(z)}-2\beta\left(\sqrt{2}-1\right)\right|+...\\+\alpha_n\left|\frac{zf_n'(z)}{f_n(z)}-2\beta\left(\sqrt{2}-1\right)\right|-\\-\sum_{i=1}^{n}2\alpha_i\beta\left(\sqrt{2}-1\right)-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}.\end{array}$$

Because $\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-2\beta\left(\sqrt{2}-1\right)\right|>0$ for all $i\in\{1,...,n\}$, obtain that

$$\sqrt{2}\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)>-\sum_{i=1}^{n}2\alpha_i\beta\left(\sqrt{2}-1\right)-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}. \tag{8}$$

Using the hypothesis (3) in (8) we have:

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)>0 \tag{9}$$

so, F is the convex function.

Corollary 2. *Let α the real numbers with the properties $0<\alpha\leq\frac{\sqrt{2}}{2\beta\left(\sqrt{2}-1\right)+\sqrt{2}}$. We suppose that the functions $f\in SH(\beta)$ and $\beta>0$. In this conditions the integral operator $F(z)=\int_0^z\left(\frac{f(z)}{z}\right)^{\alpha}dt$ is convex.*

Proof. In the Theorem 1, we consider $n=1$, $\alpha_1=\alpha$ and $f_1=f$.

References

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