INTEGRAL AVERAGES OF MULTIPLE POISSON KERNELS

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Abstract. In [3], Galbraith and Green obtained the mean value of the Poisson kernel $P(r,\theta)$ for $n > -1$. In [4], Haruki and Rassias introduced two generalizations of the Poisson kernel which are $Q(\theta;a,b)$ and $R(\theta;a,b,c,d)$, and obtained the integral value of $Q^2(\theta;a,b)$ by means of $R(\theta;a,b,c,d)$. Also, they set as an open problem the result of mean values for $n \in \mathbf{N}$. This problem is solved by Kim [5]. Then in [1], the author generalized the result of Kim and obtained the mean values of both $Q(\theta;a,b)$ and $R(\theta;a,b,c,d)$ for all real u.

The purpose of this paper is to give integral averages of multiple Poisson kernels, introduced by Spreafico [6], for all real u. Thus we generalize the main result of Spreafico.

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1. Introduction

It is well known that the Poisson kernel in two dimensions is defined by

$$P(r,\theta) \overset{def}{=} \frac{1-r^2}{(1-re^{i\theta})(1-re^{-i\theta})},$$

and the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta)d\theta = 1$$

holds. Here r is a real parameter satisfying $|r| < 1$.

In [4], Haruki and Rassias introduced two generalizations of the Poisson kernel.

The first generalization is defined by

$$Q(\theta; a, b) \stackrel{def}{=} \frac{1 - ab}{(1 - ae^{i\theta})(1 - be^{-i\theta})},$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

The second generalization is defined by

$$R(\theta; a, b, c, d) = \frac{L(a, b, c, d)}{(1 - ae^{i\theta})(1 - be^{-i\theta})(1 - ce^{i\theta})(1 - de^{-i\theta})},$$

where a, b, c, d are complex parameters satisfying $|a| < 1$, $|b| < 1$, $|c| < 1$ and $|d| < 1$ and

$$L(a, b, c, d) \stackrel{def}{=} \frac{(1 - ab)(1 - ad)(1 - bc)(1 - cd)}{1 - abcd}.$$

Then they proved the integral formulas

$$\frac{1}{2\pi} \int_0^{2\pi} Q(\theta; a, b) d\theta = 1,$$

$$\frac{1}{2\pi} \int_0^{2\pi} R(\theta; a, b, c, d) d\theta = 1.$$

If we set $c = a$ and $d = b$ in the last integral, then we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} Q(\theta; a, b)^2 d\theta = \frac{1 + ab}{1 - ab}.$$

Afterwards they set the following definition and open problem.

For $n = 0, 1, 2, \ldots$, let

$$I_n \stackrel{def}{=} \frac{1}{2\pi} \int_0^{2\pi} Q(\theta; a, b)^{n+1} d\theta,$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Open Problem. Compute I_n for $n = 2, 3, 4, \ldots$.

In [5], Kim gave a solution to this open problem. In [1], the author generalized I_n as follows.

Theorem A. [1] *For any real number u, the following holds*

$$J_u := \frac{1}{2\pi}\int_0^{2\pi} Q\left(\theta; a, b\right)^u d\theta = \left(1-ab\right)^u \; {}_2F_1\left(u, u; 1; ab\right),$$

where ${}_2F_1$ is the usual hypergeometric function.

Note that, setting $a = b = r$ generalizes the integral

$$\frac{1}{2\pi}\int_0^{2\pi} P^{n+1}(r,\theta)d\theta$$

for all real $n > -1$, ([3]).

In [1], the author also gave integral averages of $R\left(\theta; a, b, c, d\right)$ as

$$K_u := \frac{1}{2\pi}\int_0^{2\pi} R\left(\theta; a, b, c, d\right)^u d\theta = L\left(a, b, c, d\right)^u \sum_{j+l=k+m} \frac{\left(u\right)_j \left(u\right)_k \left(u\right)_l \left(u\right)_m}{j!k!l!m!} a^j b^k c^l d^m.$$

2.Multiple poisson kernels

In [6], Spreafico obtained the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} P_{a,b}\left(\theta\right) d\theta = \sum_{k=1}^{N} \frac{b_k^{N-1}}{1-a_k b_k} \prod_{n=1, n\neq k}^{N} \frac{1}{\left(1-a_n b_k\right)\left(b_k - b_n\right)},$$

where $a = (a_n)$ and $b = (b_n)$ are two vectors in the complex open unit N-ball and

$$P_{a,b}\left(\theta\right) := \prod_{n=1}^{N} \frac{1}{\left(1-a_n e^{i\theta}\right)\left(1-b_n e^{-i\theta}\right)}.$$

Let us define

$$M\left(a, b\right)^{-1} := \sum_{k=1}^{N} \frac{b_k^{N-1}}{1-a_k b_k} \prod_{n=1, n\neq k}^{N} \frac{1}{\left(1-a_n b_k\right)\left(b_k - b_n\right)}.$$

Then multiple Poisson kernel is of the form

$$P_N(\theta; a, b) := \frac{M\left(a, b\right)}{\prod\limits_{n=1}^{N}\left(1 - a_n e^{i\theta}\right)\left(1 - b_n e^{-i\theta}\right)}$$

and satisfies the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} P_N(\theta; a, b)d\theta = 1.$$

The purpose of this paper is to give integral averages of multiple Poisson kernels $P_N(\theta; a, b)$, for all real u.

Theorem 1. *For all real u, the following holds*

$$M_u := \frac{1}{2\pi}\int_0^{2\pi} P_N(\theta; a, b)^u d\theta = M\left(a, b\right)^u \sum_{l=m}\prod_{n=1}^{N}\frac{\left(u\right)_{l_n}\left(u\right)_{m_n}}{l_n! m_n!}a_n^{l_n}b_n^{m_n},$$

where

$$l := \sum_{n=1}^{N} l_n \qquad \text{and} \qquad m := \sum_{n=1}^{N} m_n.$$

Proof. Let u be any real number. Define the Pochhammer symbol by

$$\left(u\right)_k := \frac{\Gamma(u+k)}{\Gamma(u)} \qquad \left(u \neq -n, \quad n = 0, 1, 2, \ldots\right),$$

where Γ is the gamma function. If $u = -n$ is a nonpositive integer, define $\left(-n\right)_k := \left(-n\right)\left(-n+1\right)\cdots\left(-n+k-1\right)$, so that $\left(-n\right)_k = 0$ for $k = n+1, n+2, \ldots$. Then

$$\frac{1}{\left(1-w\right)^u} = \sum_{k=0}^{\infty}\frac{\left(u\right)_k}{k!}w^k \qquad \left(|w| < 1\right).$$

For $z = e^{i\theta}$, one computes that

$$\begin{aligned} M_u &= \frac{1}{2\pi}\int_0^{2\pi} P_N(\theta; a, b)^u d\theta \\ &= \frac{M\left(a, b\right)^u}{2\pi}\int_0^{2\pi}\prod_{n=1}^{N}\frac{1}{\left(1 - a_n e^{i\theta}\right)^u\left(1 - b_n e^{-i\theta}\right)^u}d\theta \end{aligned}$$

$$= \frac{M(a,b)^u}{2\pi i}\int_{|z|=1}\frac{dz}{z\prod\limits_{n=1}^{N}(1-a_n z)^u(1-b_n/z)^u}$$

$$= \frac{M(a,b)^u}{2\pi i}\int_{|z|=1}\frac{1}{z}\prod_{n=1}^{N}\left[\left(\sum_{l_n=0}^{\infty}\frac{(u)_{l_n}}{l_n!}a_n^{l_n}z^{l_n}\right)\left(\sum_{m_n=0}^{\infty}\frac{(u)_{m_n}}{m_n!}\frac{b_n^{m_n}}{z^{m_n}}\right)\right]dz.$$

Thus we get

$$M_u = M(a,b)^u\sum_{l=m}\prod_{n=1}^{N}\frac{(u)_{l_n}(u)_{m_n}}{l_n!m_n!}a_n^{l_n}b_n^{m_n},$$

where

$$l := \sum_{n=1}^{N} l_n \qquad \textit{and} \qquad m := \sum_{n=1}^{N} m_n.$$

Corollary 2. *As a consequence of Theorem* 1, *we have*
(i) $M_1 = J_1 = 1$, *for* $N = 1$ *and* $u = 1$.
(ii) $M_2 = J_2 = \frac{1+ab}{1-ab}$, *for* $N = 1$ *and* $u = 2$.
(iii) $M_1 = K_1 = 1$, *for* $N = 2$ *and* $u = 1$.
(iv) $M_1 = 1$, *for* $N = 3$ *and* $u = 1$ (*see* [2]).

2.References

[1] S. Bulut, *Integral averages of two generalizations of the Poisson kernel by Haruki and Rassias*, J. Appl. Math. Stochastic Anal. Volume 2008 (2008), Article ID 760214, 6 pages.

[2] S. Bulut, *A new generalization of the Poisson kernel*, Math. J. Okayama Univ. **48** (2006), 173-180.

[3] A. S. Galbraith and J. W. Green, *A note on the mean value of the Poisson kernel*, Bull. Amer. Math. Soc. **53** (1947), 314-320.

[4] H. Haruki and Th. M. Rassias, *New generalizations of the Poisson kernel*, J. Appl. Math. Stochastic Anal. **10** (1997), no. 2, 191-196.

[5] B. Kim, *The solution of an open problem given by H. Haruki and T. M. Rassias*, J. Appl. Math. Stochastic Anal. **12** (1999), no. 3, 261-263.

[6] M. Spreafico, *Multiple Poisson kernels*, Math. J. Okayama Univ. **51** (2009), 177-178.

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