Acta Universitatis Apulensis No. 22/2010
ISSN: 1582-5329 pp. 35-39

A NOTE ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN OPERATOR

Alina Alb Lupaş, Adriana Cătaş

Abstract. By means of a generalized Sălăgean differential operator we define a new class $\mathcal{BO}(m, \mu, \alpha, \lambda)$ involving functions $f \in \mathcal{A}_n$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

2000 *Mathematics Subject Classification*: 30C45

1. Introduction and definitions

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U, $n \in \mathbb{N} = \{1, 2, ...\}$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}^*(\alpha)$ we denote a subclass of $\mathcal{A}_n$ consisting of starlike univalent functions of order α, $0 \leq \alpha < 1$ which satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{2}$$

Further, a function f belonging to $\mathcal{S}$ is said to be convex of order α in U, if and only if

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad z \in U \tag{3}$$

for some α, $(0 \leq \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the class of functions in $\mathcal{S}$ which are convex of order α in U and denote by $\mathcal{R}(\alpha)$ the class of functions in $\mathcal{A}_n$ which satisfy

$$\mathrm{Re} f'(z) > \alpha, \quad z \in U. \tag{4}$$

It is well known that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let D^m be a generalized Sălăgean operator introduced by Al-Oboudi in [3], $D^m : \mathcal{A}_n \to \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, defined as

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda > 0 \\ D^m f(z) &= D_\lambda(D^{m-1} f(z)), \quad z \in U. \end{aligned}$$

We note that if $f \in \mathcal{A}_n$, then

$$D^m f(z) = z + \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^m a_j z^j, \quad z \in U.$$

For $\lambda = 1$, we get the Sălăgean operator [7].

To prove our main theorem we shall need the following lemma.

Lemma 1.[6] *Let p be analytic in U with $p(0) = 1$ and suppose that*

$$\mathrm{Re}\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{5}$$

Then $\mathrm{Re}p(z) > \alpha$ *for* $z \in U$ *and* $1/2 \le \alpha < 1$.

2. Main results

Definition 1. *We say that a function $f \in \mathcal{A}_n$ is in the class $\mathcal{BO}(m, \mu, \alpha, \lambda)$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \ge 0$, $\lambda \ge 0$, $\alpha \in [0, 1)$ if*

$$\left| \frac{D_\lambda^{m+1} f(z)}{z} \left(\frac{z}{D_\lambda^m f(z)} \right)^\mu - 1 \right| < 1 - \alpha \qquad z \in U. \tag{6}$$

Remark 1. The family $\mathcal{BO}(m, \mu, \alpha, \lambda)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BO}(0, 1, \alpha, 1) \equiv \mathcal{S}^*(\alpha)$, $\mathcal{BO}(1, 1, \alpha, 1) \equiv \mathcal{K}(\alpha)$ and $\mathcal{BO}(0, 0, \alpha, 1) \equiv \mathcal{R}(\alpha)$. Another interesting subclasses are the special case $\mathcal{BO}(0, 2, \alpha, 1) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [5] , the class $\mathcal{BO}(0, \mu, \alpha, 1) \equiv \mathcal{B}(\mu, \alpha)$ introduced by Frasin and Jahangiri [6] and the class $\mathcal{BO}(m, \mu, \alpha, 1)$ introduced and studied by A.Cătaş and A. Alb Lupaş [2], [4].

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BO}(m,\mu,\alpha,\lambda)$. Consequently, as a special case, we show that convex functions of order $1/2$ are also members of the above defined family.

Theorem 2. *For the function* $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geq 0$, $\lambda > 0$, $1/2 \leq \alpha < 1$ *if*

$$\frac{1}{\lambda}\frac{D_\lambda^{m+2} f(z)}{D_\lambda^{m+1} f(z)} - \frac{\mu}{\lambda}\frac{D_\lambda^{m+1} f(z)}{D_\lambda^{m} f(z)} + \frac{\mu-1}{\lambda} + 1 \prec 1 + \beta z, \quad z \in U \tag{7}$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha}$$

then $f \in \mathcal{BO}(m,\mu,\alpha,\lambda)$.

Proof. If we consider

$$p(z) = \frac{D_\lambda^{m+1} f(z)}{z}\left(\frac{z}{D_\lambda^{m} f(z)}\right)^{\mu} \tag{8}$$

then $p(z)$ is analytic in U with $p(0) = 1$. A simple differentiation yields

$$\frac{zp'(z)}{p(z)} = \frac{1}{\lambda}\frac{D_\lambda^{m+2} f(z)}{D_\lambda^{m+1} f(z)} - \frac{\mu}{\lambda}\frac{D_\lambda^{m+1} f(z)}{D_\lambda^{m} f(z)} + \frac{\mu-1}{\lambda}. \tag{9}$$

Using (7) we get

$$\operatorname{Re}\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha-1}{2\alpha}.$$

Thus, from Lemma 1 we deduce that

$$\operatorname{Re}\left\{\frac{D_\lambda^{m+1} f(z)}{z}\left(\frac{z}{D_\lambda^{m} f(z)}\right)^{\mu}\right\} > \alpha.$$

Therefore, $f \in \mathcal{BO}(m,\mu,\alpha,\lambda)$, by Definition 1.

As a consequence of the above theorem we have the following interesting corollaries.

Corollary 3.[1] *If* $f \in \mathcal{A}_n$ *and*

$$\operatorname{Re}\left\{\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in U, \tag{10}$$

then $f \in \mathcal{BO}(1,1,1/2,1)$ *hence*

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U. \tag{11}$$

That is, f *is convex of order* $\frac{1}{2}$.

Corollary 4.[1] *If* $f \in \mathcal{A}_n$ *and*

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U, \tag{12}$$

then

$$\mathrm{Re} f'(z) > \frac{1}{2}, \quad z \in U. \tag{13}$$

In another words, if the function f *is convex of order* $\frac{1}{2}$ *then* $f \in \mathcal{BO}(0,0,\frac{1}{2},1) \equiv \mathcal{R}\left(\frac{1}{2}\right)$.

Corollary 5.[1] *If* $f \in \mathcal{A}_n$ *and*

$$\mathrm{Re}\left\{\frac{f(z)+3zf'(z)+z^2f''(z)}{f'(z)+zf'(z)} - \frac{zf'(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U, \tag{14}$$

then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in U. \tag{15}$$

That is, f *is a starlike function.*

Corollary 6.[1] *If* $f \in \mathcal{A}_n$ *and*

$$\mathrm{Re}\left\{\frac{-f(z)+5zf'(z)+z^2f''(z)}{f(z)+zf(z)}\right\} > -\frac{1}{2}, \quad z \in U, \tag{16}$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{z}+f'(z)-2\right\} > 2, \quad z \in U. \tag{17}$$

References

[1] A. Alb Lupaş, *A subclass of analytic functions defined by Ruscheweyh derivative*, Acta Universitatis Apulensis, nr. 19/2009, 31-34.

[2] A.Alb Lupaş and A. Cătaş, *A note on a subclass of analytic functions defined by differential Sălăgean operator*, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, nr. 2(60), 2009, 131-134.

[3] F.M. Al-Oboudi, *On univalent functions defined by generalized Sălăgean operator*, Internat. J. Math. and Math. Sci., 27(2004), 1429-1436.

[4] A. Cătaş and A. Alb Lupaş, *On Sufficient Conditions for Certain Subclass of Analytic Functions Defined by Differential Sălăgean Operator*, International Journal of Open Problem in Complex Analysis, Vol. 1, No. 2, Nov. 2009, 14-18.

[5] B.A. Frasin and M. Darus, *On certain analytic univalent functions*, Internat. J. Math. and Math. Sci., 25(5), 2001, 305-310.

[6] B.A. Frasin and Jay M. Jahangiri, *A new and comprehensive class of analytic functions*, Analele Universităţii din Oradea, Tom XV, 2008, 61-64.

[7] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.

Alb Lupaş Alina
Department of Mathematics
University of Oradea
str. Universităţii nr. 1, 410087, Oradea, Romania
email: *dalb@uoradea.ro*

Cătaş Adriana
Department of Mathematics
University of Oradea
str. Universităţii nr. 1, 410087, Oradea, Romania
email: *acatas@uoradea.ro*