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A NEW CLASS OF HARMONIC MULTIVALENT FUNCTIONS DEFINED BY AN INTEGRAL OPERATOR

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Abstract. We define and investigate a new class of harmonic multivalent functions defined by Sălăgean integral operator. We obtain coefficient inequalities and distortion bounds for the functions in this class.

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1. Introduction

For fixed positive integer p, denote by $H(p)$ the set of all harmonic multivalent functions $f = h+\overline{g}$ which are sense-preserving in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ where h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \qquad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1. \tag{1}$$

The integral operator I^n was introduced by Sălăgean [9]. For fixed positive integer n and for $f = h + \overline{g}$ given by (1) we define the modified Sălăgean operator $I^n f$ as

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}; \quad p > n, \quad z \in \mathbb{U} \tag{2}$$

where

$$I^n h(z) = z^p + \sum_{k=2}^{\infty} \big(\frac{p}{k+p-1}\big)^n a_{k+p-1} z^{k+p-1}$$

and

$$I^n g(z) = \sum_{k=1}^{\infty} \big(\frac{p}{k+p-1}\big)^n b_{k+p-1} z^{k+p-1}.$$

It is known that, (see[3]), the harmonic function $f = h + \overline{g}$ is sense- preserving in U if $|g'| < |h'|$ in $\mathbb{U}$. The class $H(p)$ was studied by Ahuja and Jahangiri [1] and the class $H(p)$ for $p = 1$ was defined and studied by Jahangiri et al. in [6].

For fixed positive integers n, p, and for $0 \leq \alpha < 1, \quad \beta \geq 0$ we let $H_p(n+1, n, \alpha, \beta)$ denote the class of multivalent harmonic functions of the form (1) that satisfy the condition

$$\text{Re}\,\Big\{\frac{I^n f(z)}{I^{n+1} f(z)}\Big\} > \beta\Big|\frac{I^n f(z)}{I^{n+1} f(z)} - 1\Big| + \alpha. \tag{3}$$

The subclass $H_p^-(n+1, n, \alpha, \beta)$ consists of functions $f_n = h + \overline{g_n}$ in $H_p(n, \alpha, \beta)$ so that h and g are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1. \tag{4}$$

The families $H_p(n+1, n, \alpha, \beta)$ and $H_p^-(n+1, n, \alpha, \beta)$ include a variety of well-known classes of harmonic functions as well as many new ones. For example $H_1^-(1, 0, \alpha, 0) \equiv HS(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in $\mathbb{U}$, $H_1^-(2, 1, \alpha, 0) \equiv HK(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in $\mathbb{U}$ and $H_1^-(n+1, n, \alpha, 0) \equiv H^-(n, \alpha)$ is the class os Sălăgean type harmonic univalent functions.

For the harmonic functions f of the form (1) with $b_1 = 0$, Avei and Zlotkiewicz [2] showed that if

$$\sum_{k=2}^{\infty} k(|a_k| + b_k|) \leq 1,$$

then $f \in HS(0)$ and if

$$\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \leq 1,$$

then $f \in HK(0)$. Silverman [10] proved that the above two coefficient conditions are also necessary if $f = h + \overline{g}$ has negative coefficients. Later, Silverman and Silvia[11] improved the results of [6] and [9] to the case b_1 not necessarily zero.

For the harmonic functions f of the form (4) with $n = 1$, Jahangiri [5] showed that $f \in HS(\alpha)$ if and only if

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| + \sum_{k=1}^{\infty}(k+\alpha)|b_k| \leq 1 - \alpha$$

and $f \in H_1^-(2,1,\alpha,0)$ if and only if

$$\sum_{k=2}^{\infty} k(k-\alpha)|a_k| + \sum_{k=1}^{\infty} k(k+\alpha)|b_k| \leq 1-\alpha.$$

In this paper, the coefficient conditions for the classes $HS(\alpha)$ and $HK(\alpha)$ are extended to the class $H_p(n+1,n,\alpha,\beta)$, of the form (3) above. Furthermore, we determine distortion theorem for the functions in $H_p^-(n+1,n,\alpha,\beta)$.

2.Main results

In the first theorem, we introduce a sufficient coefficient bound for harmonic functions in $H_p(n+1,n,\alpha,\beta)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1). If*

$$\sum_{k=1}^{\infty}\{\Psi(n+1,n,p,\alpha,\beta)|a_{k+p-1}| + \Theta(n+1,n,p,\alpha,\beta)|b_{k+p-1}|\} \leq 2, \tag{5}$$

where

$$\Psi(n+1,n,p,\alpha,\beta) = \frac{\left(\frac{p}{k+p-1}\right)^n(1+\beta) - (\beta+\alpha)\left(\frac{p}{k+p-1}\right)^{n+1}}{1-\alpha},$$

and

$$\Theta(n+1,n,p,\alpha,\beta) = \frac{\left(\frac{p}{k+p-1}\right)^n(1+\beta) + \left(\frac{p}{k+p-1}\right)^{n+1}(\beta+\alpha)}{1-\alpha},$$

$a_p = 1, \quad 0 \leq \alpha < 1, \quad \beta \geq 0, \quad n \in \mathbb{N}$. *Then $f \in H_p(n+1,n,p,\alpha,\beta)$.*

Proof. According to (2) and (3) we only need to show that

$$\text{Re}\,\Big(\frac{I^n f(z) - \alpha I^{n+1} f(z) - \beta e^{i\Theta}|I^n f(z) - I^{n+1} f(z)|}{I^{n+1} f(z)}\Big) \geq 0.$$

The case $r = 0$ is obvious. For $0 < r < 1$ it follows that

$$\text{Re}\,\Big(\frac{I^n f(z) - \alpha I^{n+1} f(z) - \beta e^{i\theta}|I^n f(z) - I^{n+1} f(z)|}{I^{n+1} f(z)}\Big) =$$

$$= \text{Re}\,\Big\{\frac{(1-\alpha)z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}\left[\Gamma^n - \alpha\Gamma^{n+1}\right]}{z^p + \sum_{k=2}^{\infty}\Gamma^{n+1} a_{k+p-1} z^{k+p-1} + (-1)^{n+1}\sum_{k=1}^{\infty}\Gamma^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}}$$

$$+\frac{(-1)^n \sum_{k=1}^{\infty} \overline{b}_{k+p-1}\overline{z}^{k+p-1}\left[\Gamma^n+\alpha\Gamma^{n+1}\right]}{z^p+\sum_{k=2}^{\infty}\Gamma^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\Gamma^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}}-$$

$$\left.\frac{\beta e^{i\Theta}\left|\sum_{k=2}^{\infty}a_{k+p-1}z^{k+p-1}\left[\Gamma^n-\Gamma^{n+1}\right]+(-1)^n\sum_{k=1}^{\infty}\overline{b}_{k+p-1}\overline{z}^{k+p-1}\left[\Gamma^n+\Gamma^{n+1}\right]\right|}{z^p+\sum_{k=2}^{\infty}\Gamma^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\Gamma^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}}\right\}$$

$$=\mathrm{Re}\left\{\frac{1-\alpha+\sum_{k=2}^{\infty}a_{k+p-1}z^{k-1}\left[\Gamma^n-\alpha\Gamma^{n+1}\right]}{1+\sum_{k=2}^{\infty}\Gamma^{n+1}a_{k+p-1}z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\Gamma^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}\right.$$

$$+\frac{(-1)^n\sum_{k=1}^{\infty}\overline{b}_{k+p-1}\overline{z}^{k-1+p}z^{-p}\left[\Gamma^n+\alpha\Gamma^{n+1}\right]}{1+\sum_{k=2}^{\infty}\Gamma^{n+1}a_{k+p-1}z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\Gamma^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}-$$

$$\left.\frac{\beta e^{i\Theta}z^{-p}\left|\sum_{k=2}^{\infty}\left[\Gamma^n-\Gamma^{n+1}\right]a_{k+p-1}z^{k+p-1}+(-1)^n\sum_{k=1}^{\infty}\left[\Gamma^n+\Gamma^{n+1}\right]\overline{b}_{k+p-1}\overline{z}^{k+p-1}\right|}{1+\sum_{k=2}^{\infty}\Gamma^{n+1}a_{k+p-1}z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\Gamma^{n+1}\overline{b}_{k+p-1}\overline{z}^{k+p-1}z^{-p}}\right\}$$

$$=\mathrm{Re}\ \frac{(1-\alpha)+A(z)}{1+B(z)},$$

where

$$\Gamma=\frac{p}{k+p-1}.$$

For $z=re^{i\Theta}$ we have

$$A(re^{i\Theta})=\sum_{k=2}^{\infty}(A^n-\alpha A^{n+1})a_{k+p-1}r^{k-1}e^{(k-1)\Theta i}+$$

$$+(-1)^n\sum_{k=1}^{\infty}(A^n+A^{n+1}\alpha)\overline{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\Theta i}-\beta e^{-(p-1)i\Theta}\mathcal{D}(n+1,n,p,\alpha),$$

where

$$\begin{aligned}\mathcal{D}(n+1,n,p,\alpha) &= = |\sum_{k=2}^{\infty}(A^n - A^{n+1})a_{k+p-1}r^{k-1}e^{-(k+p-1)i\Theta} \\ &+ (-1)^n\sum_{k=1}^{\infty}(A^n + A^{n+1})\bar{b}_{k+p-1}r^{k-1}e^{-(k+p-1)i\Theta}|,\end{aligned}$$

and

$$\begin{aligned}B(re^{i\Theta}) &= \sum_{k=2}^{\infty}A^{n+1}a_{k+p-1}r^{k-1}e^{(k-1)\Theta i} \\ &+ (-1)^{n+1}\sum_{k=1}^{\infty}A^{n+1}\bar{b}_{k+p-1}r^{k-1}e^{-(k+2p-1)\Theta i}.\end{aligned}$$

Setting

$$\frac{1-\alpha+A(z)}{1+B(z)} = (1-\alpha)\frac{1+w(z)}{1-w(z)}.$$

The proof will be complete if we can show that $|w(z)| \leq r < 1$. This is the case since, by the condition (5), we can write:

$$|w(z)| = |\frac{A(z)-(1-\alpha)B(z)}{A(z)+(1-\alpha)B(z)+2(1-z)}| \leq$$

$$\frac{\sum_{k=1}^{\infty}[(1+\beta)(A^n - A^{n+1})|a_{k+p-1}| + (1+\beta)(A^n + A^{n+1})|b_{k+p-1}|]r^{k-1}}{4(1-\alpha)-\sum_{k=1}^{\infty}\{[A^n(1+\beta)-\Lambda A^{n+1}]|a_{k+p-1}| + [A^n(1+\beta)+\Lambda A^{n+1}]|b_{k+p-1}|\}r^{k-1}} <$$

$$< \frac{\sum_{k=1}^{\infty}(1+\beta)(A^n - A^{n+1})|a_{k+p-1}| + (A^m + A^{n+1})(1+\beta)|b_{k+p-1}|}{4(1-\alpha)-\{\sum_{k=1}^{\infty}[A^n(1+\beta)-\Lambda A^{n+1}]|a_{k+p-1}| + [A^n(1+\beta)+\Lambda A^{n+1}]|b_{k+p-1}|\}} \leq$$

$$\leq 1,$$

where $\Lambda = \beta + 2\alpha - 1$.

The harmonic univalent functions

$$f(z) = z^p + \sum_{k=2}^{\infty}\frac{1}{\Psi(n+1,n,p,\alpha,\beta)}x_k z^{k+p-1} + \sum_{k=1}^{\infty}\frac{1}{\Theta(n+1,n,p,\alpha,\beta)}\overline{y_k z^{k+p-1}}, \quad (6)$$

where $n \in \mathbb{N}, 0 \le \alpha < 1, \beta \ge 0$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (5) is sharp.

The functions of the form (6) are in $H_p(n+1, n, \alpha, \beta)$ because

$$\sum_{k=1}^{\infty} [\Psi(n+1, n, p, \alpha, \beta)|a_{k+p-1}| + \Theta(n+1, n, p, \alpha, \beta)|b_{k+p-1}|] =$$

$$= 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2.$$

In the following theorem it is show that the condition (5) is also necessary for the function $f_n = h + \overline{g}_n$, where h and g_n are of the form (4).

Theorem 2.2. *Let $f_n = h + \overline{g}_n$ be given by (4). Then $f_n \in H_p^-(n+1, n, \alpha, \beta)$ if and only if*

$$\sum_{k=1}^{\infty} [\Psi(n+1, n, p, \alpha, \beta) a_{k+p-1} + \Theta(n+1, n, p, \alpha, \beta) b_{k+p-1}] \le 2, \tag{7}$$

$a_p = 1, 0 \le \alpha < 1, n \in \mathbb{N}$.

Proof. Since $H_p^-(n+1, n, \alpha, \beta) \subset H_p(n+1, n, \alpha, \beta)$, we only need to prove the "only if" part of the theorem. For functions f_n of the form (4), we note that the condition

$$\text{Re}\, \Big\{ \frac{I^n f(z)}{I^{n+1} f(z)} \Big\} > \beta \Big| \frac{I^n f(z)}{I^{n+1} f(z)} - 1 \Big| + \alpha$$

is equivalent to

$$\text{Re}\, \Big\{ \frac{(1-\alpha) z^p - \sum_{k=2}^{\infty} (\Gamma^n - \alpha \Gamma^{n+1}) a_{k+p-1} z^{k+p-1}}{z^p - \sum_{k=2}^{\infty} \Gamma^{n+1} a_{k+p-1} z^{k+p-1} + (-1)^{2n} \sum_{k=1}^{\infty} \Gamma^{n+1} b_{k+p-1} \overline{z}^{k+p-1}} +$$

$$+ \frac{(-1)^{2n-1} \sum_{k=1}^{\infty} (\Gamma^n + \Gamma^{n+1} \alpha) b_{k+p-1} \overline{z}^{k+p-1}}{z^p - \sum_{k=2}^{\infty} \Gamma^{n+1} a_{k+p-1} z^{k+p-1} + (-1)^{2n} \sum_{k=1}^{\infty} \Gamma^{n+1} b_{k+p-1} \overline{z}^{k+p-1}}$$

$$-\frac{\beta e^{i\Theta}\left|-\sum_{k=2}^{\infty}(\Gamma^{n}-\Gamma^{n+1})a_{k+p-1}z^{k+p-1}+(-1)^{2n-1}\sum_{k=1}^{\infty}(\Gamma^{n}+\Gamma^{n+1})\overline{b}_{k+p-1}\overline{z}^{k+p-1}\right|}{z^{p}-\sum_{k=2}^{\infty}\Gamma^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{2n}\sum_{k=1}^{\infty}\Gamma^{n+1}b_{k+p-1}\overline{z}^{k+p-1}}\}$$

$$\geq 0, \tag{8}$$

where $\Gamma=\frac{p}{p+k-1}$.

The above required condition (8) must hold for all values of $z\in\mathbb{U}$. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}[\Gamma^{n}(1+\beta)-(\beta+\alpha)\Gamma^{n+1}]a_{k+p-1}r^{k-1}}{1-\sum_{k=2}^{\infty}\Gamma^{n+1}a_{k+p-1}r^{k-1}+\sum_{k=1}^{\infty}\Gamma^{n+1}b_{k+p-1}r^{k+p-1}} \tag{9}$$

$$+\frac{-\sum_{k=1}^{\infty}[\Gamma^{n}(1+\beta)+\Gamma^{n+1}(\beta+\alpha)]b_{k+p-1}r^{k-1}}{1-\sum_{k=2}^{\infty}\Gamma^{n+1}a_{k+p-1}r^{k-1}+\sum_{k=1}^{\infty}\Gamma^{n+1}b_{k+p-1}r^{k-1}}\geq 0.$$

If the condition (8) does not hold, then the expression in (9) is negative for r sufficiently close to 1. Hence there exist $z_0=r_0$ in $(0,1)$ for which the quotient in (9) is negative. This contradicts the required condition for $f_n\in H_p^{-}(n+1,n,\alpha,\beta)$. And so the proof is complete.

The following theorem gives the distortion bounds for functions in $H_p^{-}(n+1,n,\alpha,\beta)$ which yields a covering results for this class.

Theorem 2.3. *Let $f_n\in H_p^{-}(n+1,n,\alpha,\beta)$. Then for $|z|=r<1$ we have*

$$|f_n(z)|\leq(1+b_p)r^p+[\Phi(n+1,n,p,\alpha,\beta)-\Omega(n+1,n,p,\alpha,\beta)b_p]r^{n+1+p}$$

and

$$|f_n(z)|\geq(1-b_p)r^p-\{\Phi(n+1,n,p,\alpha,\beta)-\Omega(n+1,n,p,\alpha,\beta)b_p\}r^{n+p+1}$$

where,

$$\Phi(n+1,n,p,\alpha,\beta)=\frac{1-\alpha}{\left(\frac{p}{p+1}\right)^{n}(1+\beta)-\left(\frac{p}{p+1}\right)^{n+1}(\beta+\alpha)},$$

$$\Omega(n+1,n,p,\alpha,\beta)=\frac{(1+\beta)+(\alpha+\beta)}{\left(\frac{p}{p+1}\right)^{n}(1+\beta)-\left(\frac{p}{p+1}\right)^{n+1}(\beta+\alpha)}.$$

Proof. We prove the right side inequality for $|f_n|$. The proof for the left hand inequality can be done using similar arguments. Let $f_n\in H_p^-(n+1,n,\alpha,\beta)$. Taking the absolute value of f_n then by Theorem 2.2, we can obtain:

$$|f_n(z)|=|z^p-\sum_{k=2}^{\infty}a_{k+p-1}z^{k+p-1}+(-1)^{n-1}\sum_{k=1}^{\infty}b_{k+p-1}\overline{z}^{k+p-1}|\leq$$

$$\leq r^p+\sum_{k=2}^{\infty}a_{k+p-1}r^{k+p-1}+\sum_{k=1}^{\infty}b_{k+p-1}r^{k+p-1}=$$

$$=r^p+b_pr^p+\sum_{k=2}^{\infty}(a_{k+p-1}+b_{k+p-1})r^{k+p-1}\leq$$

$$\leq r^p+b_pr^p+\sum_{k=2}^{\infty}(a_{k+p-1}+b_{k+p-1})r^{p+1}=$$

$$=(1+b_p)r^p+\Phi(n+1,n,p,\alpha,\beta)\sum_{k=2}^{\infty}\frac{1}{\Phi(n+1,n,p,\alpha,\beta)}(a_{k+p-1}+b_{k+p-1})r^{p+1}\leq$$

$$\leq(1+b_p)r^p+\Phi(n+1,n,p,\alpha,\beta)r^{n+p+1}\times$$

$$\times[\sum_{k=2}^{\infty}\Psi(n+1,n,p,\alpha,\beta)a_{k+p-1}+\Theta(n+1,n,p,\alpha,\beta)b_{k+p-1}]\leq$$

$$\leq(1+b_p)r^p+[\Phi(n+1,n,p,\alpha,\beta)-\Omega(n+1,n,p,\alpha,\beta)b_p]r^{n+1+p}.$$

The following covering result follows from the left hand inequality in Theorem 2.3.

Corollary 2.4. *Let $f_n\in H_p^-(n+1,n,\alpha,\beta)$, then for $|z|=r<1$ we have*

$$\{w:|w<1-b_p-[\Phi(n+1,n,p,\alpha,\beta)-\Omega(n+1,n,p,\alpha,\beta)b_p]\subset f_n(\mathbb{U})\}.$$

For $\beta=0$ we obtain the results given in [4].

For $\beta=0,p=1$ and using the differential Sălăgean operator we obtain the results given [7].

The beautiful results, for harmonic functions, was obtained by P. T. Mocanu in [8].

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