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COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS INVOLVING SALAGEAN OPERATOR

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Abstract. The main purpose of this paper is to derive some coefficient inequalities for certain classes of analytic functions which are defined by means of the Salagean operator. Relevant connections of the results presented here with those given in earlier works are also pointed out.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U}), \tag{1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ whose members are analytic and univalent functions in $\mathbb{U}$. For $0 \leqq \beta < 1$, We denote by $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ the usual subclasses of $\mathcal{A}$ consisting of functions which are, respectively, *starlike of order* β and *convex of order* β in $\mathbb{U}$.

Salagean [1] once introduced the following operator which called the Salagean operator:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N} := \{1, 2, \ldots\}).$$

We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{2}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}_{m,n}(\alpha, \beta)$ if it satisfies the inequality:

$$\Re\left(\frac{D^m f(z)}{D^n f(z)}\right) > \alpha \left|\frac{D^m f(z)}{D^n f(z)} - 1\right| + \beta \quad (\alpha \geqq 0,\ 0 \leqq \beta < 1;\ m \in \mathbb{N},\ n \in \mathbb{N}_0). \tag{3}$$

Also let $\mathcal{M}^s_{m,n}(\alpha, \beta)(s \in \mathbb{N}_0)$ be the subclass of $\mathcal{A}$ consisting of functions f which are satisfied the condition:

$$f \in \mathcal{M}^s_{m,n}(\alpha, \beta) \Longleftrightarrow D^s f \in \mathcal{N}_{m,n}(\alpha, \beta). \tag{4}$$

It is easy to see that if $s = 0$, then $\mathcal{M}^0_{m,n}(\alpha, \beta) \equiv \mathcal{N}_{m,n}(\alpha, \beta)$. We observe that, by specializing the parameters m, n, α and β, we obtain the following subclasses studied by various authors.

(1) $\mathcal{N}_{1,0}(0, \beta) \equiv \mathcal{S}^*(\beta)$ and $\mathcal{N}_{2,1}(0, \beta) \equiv \mathcal{K}(\beta)$ (see Silverman [2]).

(2) $\mathcal{N}_{1,0}(\alpha, \beta) \equiv \mathcal{S}D(\alpha, \beta)$ and $\mathcal{N}_{2,1}(\alpha, \beta) \equiv \mathcal{K}D(\alpha, \beta)$ (see Shams *et al.* [3], Owa *et al.* [4]).

(3) $\mathcal{N}_{m,n}(0, \beta) \equiv \mathcal{K}_{m,n}(\beta)$ and $\mathcal{M}^s_{m,n}(0, \beta) \equiv \mathcal{M}^s_{m,n}(\beta)$ (see Eker and Owa [5]).

The function classes $\mathcal{N}_{m,n}(\alpha, \beta)$ and $\mathcal{M}^s_{m,n}(\alpha, \beta)$ were introduced and investigated by Eker and Owa [6] (see also Srivastava and Eker [7]). They obtained many interesting results associated with them. In this paper, we aim at proving some coefficient inequalities for the function classes $\mathcal{N}_{m,n}(\alpha, \beta)$ and $\mathcal{M}^s_{m,n}(\alpha, \beta)$. Relevant connections of the results presented here with those given in earlier works are also pointed out.

2. Main Results

In order to prove our main results, we need the following lemma.

Lemma 1. *Let $\delta > 0$, $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Suppose also that the sequence $\{B_k\}_{k=1}^{\infty}$ is defined by*

$$B_1 = 1, \qquad B_2 = \frac{\delta}{|2^m - 2^n|},$$

and

$$B_k = \frac{\delta}{|k^m - k^n|}\sum_{j=1}^{k-1} j^n B_j \qquad (k \in \mathbb{N} \setminus \{1,2\}). \tag{5}$$

Then

$$B_k = \frac{\delta}{|k^m - k^n|}\prod_{j=2}^{k-1}\left(1 + \frac{\delta j^n}{|j^m - j^n|}\right) \qquad (k \in \mathbb{N} \setminus \{1,2\}). \tag{6}$$

Proof. We make use of the principle of mathematical induction to prove the assertion (6).

For $k = 3$, we know that

$$B_3 = \frac{\delta}{|3^m - 3^n|}(1 + 2^n B_2) = \frac{\delta}{|3^m - 3^n|}\prod_{j=2}^{2}\left(1 + \frac{\delta j^n}{|j^m - j^n|}\right),$$

which implies that (6) holds for $k = 3$.

We now suppose that (6) holds for $k = 3, 4, \cdots, r$. Then

$$B_r = \frac{\delta}{|r^m - r^n|}\prod_{j=2}^{r-1}\left(1 + \frac{\delta j^n}{|j^m - j^n|}\right). \tag{7}$$

Combining (5) and (7), we find that

$$\begin{aligned}
B_{r+1} &= \frac{\delta}{|(r+1)^m - (r+1)^n|}\sum_{j=1}^{r} j^n B_j \\
&= \frac{\delta}{|(r+1)^m - (r+1)^n|}\sum_{j=1}^{r-1} j^n B_j + \frac{\delta r^n}{|(r+1)^m - (r+1)^n|} B_r \\
&= \frac{\delta}{|(r+1)^m - (r+1)^n|}\frac{|r^m - r^n|}{\delta} B_r + \frac{\delta r^n}{|(r+1)^m - (r+1)^n|} B_r \\
&= \frac{|r^m - r^n| + \delta r^n}{|(r+1)^m - (r+1)^n|}\frac{\delta}{|r^m - r^n|}\prod_{j=2}^{r-1}\left(1 + \frac{\delta j^n}{|j^m - j^n|}\right) \\
&= \frac{\delta}{|(r+1)^m - (r+1)^n|}\left(1 + \frac{\delta r^n}{|r^m - r^n|}\right)\prod_{j=2}^{r-1}\left(1 + \frac{\delta j^n}{|j^m - j^n|}\right) \\
&= \frac{\delta}{|(r+1)^m - (r+1)^n|}\prod_{j=2}^{r}\left(1 + \frac{\delta j^n}{|j^m - j^n|}\right),
\end{aligned}$$

which shows that (6) holds for $k = r + 1$. The proof of Lemma 1 is evidently completed. □

Theorem 1. *If $f \in \mathcal{N}_{m,n}(\alpha, \beta)$, then*

$$|a_2| \leqq \frac{2(1-\beta)}{|1-\alpha|\,|2^m - 2^n|}, \tag{8}$$

and

$$|a_k| \leqq \frac{2(1-\beta)}{|1-\alpha|\,|k^m - k^n|} \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)j^n}{|1-\alpha|\,|j^m - j^n|}\right) \qquad (k \in \mathbb{N} \setminus \{1,2\}). \tag{9}$$

Proof. Let $f \in \mathcal{N}_{m,n}(\alpha, \beta)$ and suppose that

$$p(z) := \frac{(1-\alpha)\frac{D^m f(z)}{D^n f(z)} - (\beta - \alpha)}{1-\beta} \qquad (z \in \mathbb{U})\,. \tag{10}$$

Then p is analytic in $\mathbb{U}$ with

$$p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$

We now set

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots\,. \tag{11}$$

Combining (10) and (11), we find that

$$D^m f(z) = D^n f(z) \left(1 + \frac{1-\beta}{1-\alpha} \sum_{k=1}^{\infty} p_k z^k\right),$$

which implies that

$$(k^m - k^n)a_k = \frac{1-\beta}{1-\alpha}\left[p_{k-1} + 2^n a_2 p_{k-2} + \cdots + (k-1)^n a_{k-1} p_1\right] \quad (k \in \mathbb{N}\backslash\{1\}). \tag{12}$$

On the other hand, it is well known that

$$|p_k| \leqq 2 \qquad (k \in \mathbb{N}). \tag{13}$$

Combining (12) and (13), we obtain

$$|a_k| \leqq \frac{2(1-\beta)}{|1-\alpha|\,|k^m - k^n|} \sum_{j=1}^{k-1} j^n\,|a_j| \qquad (a_1 = 1;\; k \in \mathbb{N} \setminus \{1\}). \tag{14}$$

Suppose that $\delta = \frac{2(1-\beta)}{|1-\alpha|} > 0$, $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. We define the sequence $\{B_k\}_{k=1}^{\infty}$ by

$$B_1 = 1, \qquad B_2 = \frac{\delta}{|2^m - 2^n|},$$

and

$$B_k = \frac{\delta}{|k^m - k^n|} \sum_{j=1}^{k-1} j^n B_j \qquad (k \in \mathbb{N} \setminus \{1, 2\}). \tag{15}$$

In order to prove that

$$|a_k| \leqq B_k \qquad (k \in \mathbb{N} \setminus \{1\}),$$

we use the principle of mathematical induction. By noting that

$$|a_2| \leqq \frac{2(1-\beta)}{|1-\alpha|\,|2^m - 2^n|} = B_2. \tag{16}$$

Thus, assuming that

$$|a_k| \leqq B_k \qquad (k \in \{2, 3, \ldots, r\}),$$

we find from (14) and (15) that

$$\begin{aligned} |a_{r+1}| &\leqq \frac{2(1-\beta)}{|1-\alpha|\,|(r+1)^m - (r+1)^n|} \sum_{j=1}^{r} j^n\,|a_j| \\ &\leqq \frac{\delta}{|(r+1)^m - (r+1)^n|} \sum_{j=1}^{r} j^n B_j = B_{r+1}. \end{aligned}$$

Therefore, we have

$$|a_k| \leqq B_k \qquad (k \in \mathbb{N} \setminus \{1\}) \tag{17}$$

as desired.

By virtue of Lemma 1 and (15), we know that

$$B_k = \frac{2(1-\beta)}{|1-\alpha|\,|k^m - k^n|} \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)j^n}{|1-\alpha|\,|j^m - j^n|}\right) \qquad (k \in \mathbb{N} \setminus \{1, 2\}). \tag{18}$$

Combining (16), (17) and (18), we readily arrive at the coefficient inequalities (8) and (9) asserted by Theorem 1. □

By virtue of Theorem 1 and (4), we get the following result.

Corollary 1. *If* $f(z) \in \mathcal{M}^{s}_{m,n}(\alpha,\beta)$*, then*

$$|a_2| \leqq \frac{2(1-\beta)}{|1-\alpha|\,|2^{m+s}-2^{n+s}|},$$

and

$$|a_k| \leqq \frac{2(1-\beta)}{|1-\alpha|\,|k^{m+s}-k^{n+s}|}\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha|\,|j^m-j^n|}\right) \qquad (k\in\mathbb{N}\setminus\{1,2\}).$$

Remark 1. By specializing the parameters m and n, we get the corresponding results obtained by Owa *et al.* [4].

By means of Theorem 1 and Corollary 1, we easily get the following distortion theorems.

Corollary 2. *If* $f \in \mathcal{N}_{m,n}(\alpha,\beta)$*, then*

$$\max\left\{0,\ |z| - \frac{2(1-\beta)}{|1-\alpha|}\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha||j^m-j^n|}\right)}{|k^m-k^n|}\,|z|^k\right\}$$
$$\leqq |f(z)| \leqq |z| + \frac{2(1-\beta)}{|1-\alpha|}\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha||j^m-j^n|}\right)}{|k^m-k^n|}\,|z|^k,$$

and

$$\max\left\{0,\ 1 - \frac{2(1-\beta)}{|1-\alpha|}\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha||j^m-j^n|}\right)}{|k^{m-1}-k^{n-1}|}\,|z|^{k-1}\right\}$$
$$\leqq \left|f'(z)\right| \leqq 1 + \frac{2(1-\beta)}{|1-\alpha|}\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha||j^m-j^n|}\right)}{|k^{m-1}-k^{n-1}|}\,|z|^{k-1}.$$

Corollary 3. *If* $f \in \mathcal{M}^{s}_{m,n}(\alpha,\beta)$*, then*

$$\max\left\{0, |z| - \frac{2(1-\beta)}{|1-\alpha|}\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha||j^m-j^n|}\right)}{|k^{m+s}-k^{n+s}|}\,|z|^k\right\}$$
$$\leqq |f(z)| \leqq |z| + \frac{2(1-\beta)}{|1-\alpha|}\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha||j^m-j^n|}\right)}{|k^{m+s}-k^{n+s}|}\,|z|^k,$$

and

$$\max\left\{0, 1-\frac{2(1-\beta)}{|1-\alpha|}\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha||j^m-j^n|}\right)}{|k^{m+s-1}-k^{n+s-1}|}|z|^{k-1}\right\}$$
$$\leqq \left|f'(z)\right| \leqq 1+\frac{2(1-\beta)}{|1-\alpha|}\sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)j^n}{|1-\alpha||j^m-j^n|}\right)}{|k^{m+s-1}-k^{n+s-1}|}|z|^{k-1}.$$

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