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SOME FAMILIES OF UNIVALENT FUNCTIONS ASSOCIATED WITH SALAGEAN DERIVATIVE OPERATOR

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Abstract. Making use of Salagean derivative operator, the authors have introduced and studied new subclass $T^{\lambda}_{n,k}(\alpha,\beta,A,B)$ of normalized and univalent functions in unit disk $U=\{z:|z|<1\}$. Among other results we have established certain characterization of $T^{\lambda}_{n,k}(\alpha,\beta,A,B)$. Finally, several applications involving an integral operator and fractional calculus operators are also determined.

Keywords. Salagean operator, Hadmard product, integral operator, fractional calculus operator.

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1. Introduction and Definitions

Let A_k denote the class of functions of the form

$$f(z)=z+\sum_{j=k+1}^{\infty}a_jz^j \quad (k\in\mathbb{N}:=\{1,2,3,\dots\}), \tag{1.1}$$

which are analytic in open unit disk

$$U=\{z:z\in\mathbb{C} \quad \text{and} \quad |z|<1\}.$$

Definition 1 [6]. *We define the operator $D^n:A_k\to A_k,(n\in N_0:=\mathbb{N}\cup\{0\})$ by*

$$\begin{aligned} D^0f(z) &= f(z),\\ D^1f(z) &= zf'(z),\\ D^nf(z) &= D(D^{n-1}f(z)). \end{aligned}$$

The operator D^n is known as the Salagean derivative operator.

For the function $f(z)$ given by (1.1), it follows form above definition that

$$D^n f(z) = z + \sum_{j=k+1}^{\infty} j^n a_j z^j \quad (n \in N_0) \tag{1.2}$$

with the help of the operator D^n we define, the subclass denoted by $A_{n,k}^{\lambda}(\alpha, \beta, A, B)$ as follows.

Definition 2. *We define the class $A_{n,k}^{\lambda}(\alpha, \beta, A, B)$ by*

$$A_{n,k}^{\lambda}(\alpha, \beta, A, B) = \left\{ f \in A_k : \left| \frac{F_{n,\lambda}(z) - 1}{BF_{n,\lambda}(z) - [B + (A - B)(1 - \alpha)]} \right| < \beta \right\} \tag{1.3}$$

$(z \in U;\ n \in N_0;\ 0 \le \lambda \le 1;\ 0 \le \alpha < 1;\ 0 < \beta \le 1;\ -1 \le A < B \le 1;\ 0 \le B \le 1)$

where, for convenience,

$$F_{n,\lambda}(z) = \frac{(1 - \lambda) z (D^n f(z))' + \lambda z (D^{n+1} f(z))'}{(1 - \lambda) D^n f(z) + \lambda D^{n+1} f(z)} = \frac{\phi_{n,\lambda}(z)}{\psi_{n,\lambda}(z)}.$$

Let T_k denote the subclass of A_k consisting of functions of the form:

$$f(z) = z - \sum_{j=k+1}^{\infty} a_j z^j \quad (a_j \ge 0; j = k+1, k+2, k+3, \ldots; \quad k \in \mathbb{N}) \tag{1.4}$$

and

$$T_{n,k}^{\lambda}(\alpha, \beta, A, B) = A_{n,k}^{\lambda}(\alpha, \beta, A, B) \cap T_k. \tag{1.5}$$

We note that, in view of above definition of the class $T_{n,k}^{\lambda}(\alpha, \beta, A, B)$, specifying the parameters $k, \lambda, \alpha, \beta, A, B$ and n, we can obtain following subclasses studied by various authors.

(i) $T_{0,1}^{0}(\alpha, 1, -1, 1) = T^*(\alpha)$ and $T_{0,1}^{1}(\alpha, 1, -1, 1) = T_{1,1}^{0}(\alpha, 1, -1, 1) = C(\alpha)$ (Silverman [8]),

(ii) $T_{0,k}^{0}(\alpha, 1, -1, 1) = T_{\alpha}(k)$ and $T_{0,k}^{1}(\alpha, 1, -1, 1) = T_{1,k}^{0}(\alpha, 1, -1, 1) = C_{\alpha}(k)$ (Chatterjea [4] and Srivastava [9]),

(iii) $T_{0,k}^{\lambda}(\alpha, 1, -1, 1) = P(k, \lambda, \alpha)$ (Altintas [1]),

(iv) $T^{\lambda}_{n,k}(\alpha, 1, -1, 1) = P(k, \lambda, \alpha, n)$ (Aouf and Srivastva [3]),

(v) $T^{\lambda}_{n,k}(\alpha, \beta, -1, B) = T^{\lambda}_{n,k}(\alpha, \beta, B)$ (Srivastava, Patel and Sahoo [10]).

We have established several general properties such as coefficient inequality, distortion, inclusion properties and other related properties for aforementioned class $T^{\lambda}_{n,k}(\alpha, \beta, A, B)$.

2. Coefficient Inequalities

In this section, we provide a necessary and sufficient condition for a function f in T_k to be in $T^{\lambda}_{n,k}(\alpha, \beta, A, B)$.

Theorem 1. *Let the function f be defined by (1.4). Then $f \in T^{\lambda}_{n,k}(\alpha, \beta, A, B)$ if and only if*

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}a_j \leq \beta(B-A)(1-\alpha). \quad (2.1)$$

The result (2.1) is sharp.

Proof. Assume that the inequality (2.1) holds true. Then for $|z| = r < 1$, we observe that

$$|\phi_{n,\lambda}(z) - \psi_{n,\lambda}(z)| - \beta|B\phi_{n,\lambda}(z) - \{B + (A-B)(1-\alpha)\}\psi_{n,\lambda}(z)|$$

$$= \left| -\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j z^{j-1} \right|$$

$$-\beta \left| (B-A)(1-\alpha) - \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{-A(1-\alpha)+(j-\alpha)B\}a_j z^{j-1} \right|$$

$$\leq \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j$$

$$-\beta \left[(B-A)(1-\alpha) - \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{-A(1-\alpha)+(j-\alpha)B\}a_j \right]$$

$$\leq \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}a_j - \beta(B-A)(1-\alpha) \leq 0$$

where we have used (2.1). Hence by Maximum Modulus Theorem $f \in T_{n,k}^{\lambda}(\alpha, \beta, A, B)$.

Conversely we assume that $f \in T_{n,k}^{\lambda}(\alpha, \beta, A, B)$, then

$$\left| \frac{F_{n,\lambda}(z) - 1}{BF_{n,\lambda}(z) - [B + (A-B)(1-\alpha)]} \right|$$
$$= \left| \frac{-\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j z^{j-1}}{(B-A)(1-\alpha) - \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{-A(1-\alpha)+(j-\alpha)B\}a_j z^{j-1}} \right| < \beta,$$
$z \in U$

Since $|\Re(z)| \leq |z|$ for all z, we obtain the inequality,

$$\Re\left(\frac{\sum\limits_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j z^{j-1}}{(B-A)(1-\alpha) - \sum\limits_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{-A(1-\alpha)+(j-\alpha)B\}a_j z^{j-1}} \right) < \beta \tag{2.2}$$

Now we choose value of z on real axis so that $F_{n,\lambda}(z)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through real values. We deduce that

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j$$
$$\leq \beta(B-A)(1-\alpha) - \beta \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{-A(1-\alpha)+(j-\alpha)B\}a_j.$$

Thus

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}a_j \leq \beta(B-A)(1-\alpha).$$

Finally we note that the function f given by

$$f(z) = z - \frac{\beta(B-A)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}} z^j \tag{2.3}$$

is an extremal function for the assertion of Theorem 1.

Corollary 1. *If* $f \in T^{\lambda}_{n,k}(\alpha, \beta, A, B)$ *then*

$$a_j \leq \frac{\beta(B-A)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}} \quad (j \geq k+1; k \in \mathbb{N}). \tag{2.4}$$

The result (2.4) is sharp and the extremal functions are given by (2.3).

Remark 1. Since $1-\lambda+\lambda j \leq 1-\upsilon+\upsilon j$ for $0 \leq \lambda \leq \upsilon \leq 1 (j \geq k+1; k \in \mathbb{N})$ we have,

$$T^{\upsilon}_{n,k}(\alpha, \beta, A, B) \subseteq T^{\lambda}_{n,k}(\alpha, \beta, A, B) \quad (0 \leq \lambda \leq \upsilon \leq 1).$$

Furthermore, for $0 \leq \alpha_1 \leq \alpha_2 < 1$, we obtain

$$T^{\lambda}_{n,k}(\alpha_2, \beta, A, B) \subseteq T^{\lambda}_{n,k}(\alpha_1, \beta, A, B) \quad (0 \leq \alpha_1 \leq \alpha_2 < 1).$$

Theorem 2. *For each* $n \in N_0$,

$$T^{\lambda}_{n+1,k}(\alpha, \beta, A, B) \subset T^{\lambda}_{n,k}(\xi, \beta, A, B),$$

where

$$\xi = \frac{(1+\beta B)(k+\alpha)+\beta(B-A)(1-\alpha)}{(1+\beta B)(k+1)+\beta(B-A)(1-\alpha)} \tag{2.5}$$

The result (2.5) is sharp.

Proof. Suppose $f \in T^{\lambda}_{n+1,k}(\alpha, \beta, A, B)$. Then by Theorem 1,

$$\sum_{j=k+1}^{\infty} j^{n+1}(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}a_j \leq \beta(B-A)(1-\alpha) \tag{2.6}$$

To prove that $f \in T^{\lambda}_{n,k}(\xi, \beta, A, B)$, it is sufficient to find the largest ξ such that

$$\sum_{j=k+1}^{\infty} j^{n}(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\xi)\}a_j \leq \beta(B-A)(1-\xi). \tag{2.7}$$

Equation (2.7) is true if

$$\frac{(j-1)(1+\beta B)+\beta(B-A)(1-\xi)}{1-\xi} \leq \frac{j[(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)]}{1-\alpha} \quad (j \geq k+1; k \in \mathbb{N}),$$

that is, if

$$\xi \leq \frac{(1+\beta B)(j-1+\alpha)+\beta(B-A)(1-\alpha)}{(1+\beta B)j+\beta(B-A)(1-\alpha)} \quad (j \geq k+1; k \in \mathbb{N}). \tag{2.8}$$

Since the right hand side of (2.8) is an increasing function of j, letting $j=k+1$ in (2.8), we obtain

$$\xi \leq \frac{(1+\beta B)(k+\alpha)+\beta(B-A)(1-\alpha)}{(1+\beta B)(k+1)+\beta(B-A)(1-\alpha)}.$$

Finally, the function f given by

$$f(z) = z - \frac{\beta(B-A)(1-\alpha)}{(k+1)^n(1+\lambda k)\{k(1+\beta B)+\beta(B-A)(1-\alpha)\}} z^{k+1} \tag{2.9}$$

is an extremal function for Theorem 2.

Remark 2. Since $\xi > \alpha$, it follows from Remark 1 that

$$T^{\lambda}_{n,k}(\xi,\beta,A,B) \subset T^{\lambda}_{n,k}(\alpha,\beta,A,B) \quad (n \in N_0)$$

and hence that

$$T^{\lambda}_{n+1,k}(\alpha,\beta,A,B) \subset T^{\lambda}_{n,k}(\xi,\beta,A,B) \subset T^{\lambda}_{n,k}(\alpha,\beta,A,B) \quad (n \in N_0).$$

Theorem 3. *Let* $0 \leq \alpha_j < 1 (j=1,2)$ *and* $0 < \beta_j \leq 1 (j=1,2)$. *Then*

$$T^{\lambda}_{n,k}(\alpha_1,\beta_1,-1,B_1) = T^{\lambda}_{n,k}(\alpha_2,\beta_2,-1,B_2) \quad (n \in N_0) \tag{2.10}$$

if and only if

$$\frac{\beta_1(B_1+1)(1-\alpha_1)}{1+\beta_1 B_1} = \frac{\beta_2(B_2+1)(1-\alpha_2)}{1+\beta_2 B_2}. \tag{2.11}$$

In particular, if $0 \leq \alpha < 1$ *and* $0 < \beta \leq 1$, *then*

$$\begin{aligned} T^{\lambda}_{n,k}(\alpha,\beta,-1,B) &= T^{\lambda}_{n,k}\left(\frac{1-\beta+\beta(B+1)\alpha}{1+\beta B},1,-1,1\right) \\ &= p\left(k,\lambda,\frac{1-\beta+\beta(B+1)\alpha}{1+\beta B},n\right) \end{aligned} \tag{2.12}$$

Proof. Suppose $f \in T^{\lambda}_{n,k}(\alpha_1, \beta_1, -1, B_1)$ and let the condition (2.11) hold true. Then

$$\sum_{j=k+1}^{\infty} \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta_2 B_2)+\beta_2(B_2+1)(1-\alpha_2)\}}{\beta_2(B_2+1)(1-\alpha_2)} a_j$$

$$= \sum_{j=k+1}^{\infty} \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta_1 B_1)+\beta_1(B_1+1)(1-\alpha_1)\}}{\beta_1(B_1+1)(1-\alpha_1)} a_j \leq 1,$$

which shows that $f \in T^{\lambda}_{n,k}(\alpha_2, \beta_2, -1, B_2)$, by Theorem 1. Reversing the above steps, we can similarly prove that, under the condition (2.11),

$$f \in T^{\lambda}_{n,k}(\alpha_2, \beta_2, -1, B_2) \Rightarrow f \in T^{\lambda}_{n,k}(\alpha_1, \beta_1, -1, B_1).$$

Conversely, the assertion (2.10) can easily be shown to imply the condition (2.11). Also observe that (2.12) is a special case of (2.10) when,

$$\alpha_1 = \alpha, \quad \beta_1 = \beta, \quad B_1 = B, \quad \beta_2 = 1, \quad B_2 = 1.$$

Remark 2. For $B_1 = 1$ and $B_2 = 1$ the result of Theorem 3 was obtained by Srivastava, Patel and Sahoo [10].

Similarly we can prove following theorem.

Theorem 4. *Let* $0 \leq \alpha < 1, 0 < \beta_j \leq 1, -1 \leq A_j < B_j \leq 1$ *and* $0 \leq B_j \leq 1 (j = 1, 2)$. *Then*

$$T^{\lambda}_{n,k}(\alpha, \beta_1, A_1, B_1) = T^{\lambda}_{n,k}(\alpha, \beta_2, A_2, B_2) \quad (n \in N_0) \tag{2.13}$$

if and only if

$$\frac{\beta_1(B_1 - A_1)}{1+\beta_1 B_1} = \frac{\beta_2(B_2 - A_2)}{1+\beta_2 B_2}. \tag{2.14}$$

In particular, if $0 < \beta \leq 1, -1 \leq A < B \leq 1,$ *and* $0 \leq B \leq 1$ *then*

$$T^{\lambda}_{n,k}(\alpha, \beta, A, B) = T^{\lambda}_{n,k}(\alpha, \beta, -1, \frac{B-A-1-\beta B}{1+\beta A}) \quad (n \in N_0). \tag{2.15}$$

3. Inclusion Properties Associated with Modified Hadmard Products

Let f be defined by (1.4) and let

$$g(z) = z - \sum_{j=k+1}^{\infty} b_j z^j \quad (b_j \geq 0; j = k+1, k+2, k+3, \ldots; \quad k \in \mathbb{N}) \tag{3.1}$$

Then the modified Hadmard product (or convolution) of f and g is defined by

$$(f * g)(z) = z - \sum_{j=k+1}^{\infty} a_j b_j z^j \tag{3.2}$$

$$(a_j \geq 0;\ b_j \geq 0;\ j = k+1, k+2, k+3, \ldots; \quad k \in \mathbb{N}).$$

Theorem 5. *Let the function f defined by (1.4) and the function g defined by (3.1) belong to the class $T_{n,k}^{\lambda}(\alpha, \beta, A, B)$. Then the modified Hadmard product $f * g$ defined by (3.2) belongs to the class $T_{n,k}^{\lambda}(\eta, \beta, A, B)$, where*

$$\eta = \frac{(k+1)^n(1+\lambda k)\{(1+\beta B)k + \beta(B-A)(1-\alpha)\}^2 - \beta(B-A)(1-\alpha)^2\{(1+\beta B)k + \beta(B-A)\}}{(k+1)^n(1+\lambda k)\{(1+\beta B)k + \beta(B-A)(1-\alpha)\}^2 - \{\beta(B-A)(1-\alpha)\}^2}$$

This result is sharp.

Proof. Suppose $f, g \in T_{n,k}^{\lambda}(\alpha, \beta, A, B)$. Then we need to find largest η such that

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\eta)\}a_j b_j \leq \beta(B-A)(1-\eta) \tag{3.3}$$

Since

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}a_j \leq \beta(B-A)(1-\alpha)$$

and

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}b_j \leq \beta(B-A)(1-\alpha)$$

by the Cauchy – Schwarz inequality, we have

$$\sum_{j=k+1}^{\infty} \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{\beta(B-A)(1-\alpha)}\sqrt{a_j b_j} \leq 1 \tag{3.4}$$

Thus it is sufficient to show that

$$\frac{(j-1)(1+\beta B)+\beta(B-A)(1-\eta)}{(1-\eta)}a_jb_j$$
$$\leq \frac{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)}{(1-\alpha)}\sqrt{a_jb_j} \quad (j \geq k+1; k \in \mathbb{N}),$$

That is,

$$\sqrt{a_jb_j} \leq \frac{(1-\eta)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{(1-\alpha)\{(j-1)(1+\beta B)+\beta(B-A)(1-\eta)\}} \quad (j \geq k+1; k \in \mathbb{N}).$$

Since (3.4) implies that

$$\sqrt{a_jb_j} \leq \frac{\beta(B-A)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}} \quad (j \geq k+1; k \in \mathbb{N}),$$

Thus we have to show that

$$\frac{\beta(B-A)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}$$
$$\leq \frac{(1-\eta)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{(1-\alpha)\{(j-1)(1+\beta B)+\beta(B-A)(1-\eta)\}} \quad (j \geq k+1; k \in \mathbb{N}).$$

Or, equivalently

$$\eta \leq \frac{\begin{array}{c} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}^2 \\ -\beta(B-A)(1-\alpha)^2\{(j-1)(1+\beta B)+\beta(B-A)\} \end{array}}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}^2-\{\beta(B-A)(1-\alpha)\}^2}$$
$$(j \geq k+1; k \in \mathbb{N}) \tag{3.5}$$

Since the right hand side of (3.5) is an increasing function of j, by letting $j = k+1$ in (3.5), we obtain

$$\eta \leq \frac{\begin{array}{c} (k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}^2 \\ -\beta(B-A)(1-\alpha)^2\{(1+\beta B)k+\beta(B-A)\} \end{array}}{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}^2-\{\beta(B-A)(1-\alpha)\}^2}$$

which proves the main assertion of Theorem 5.

The sharpness of the result follows if we take

$$f(z)=g(z)=z-\frac{\beta(B-A)(1-\alpha)}{(k+1)^n(1+\lambda k)\{k(1+\beta B)+\beta(B-A)(1-\alpha)\}}z^{k+1}. \tag{3.6}$$

Theorem 6. *Let the function f and g belongs to the class $T^{\lambda}_{n,k}(\alpha,\beta,A,B)$. Then the modified Hadmard product $f*g$ belongs to the class $T^{\lambda}_{n,k}(\rho,1,-1,1)$ or equivalently, $p(k,\lambda,\rho,n)$, where*

$$\rho = \frac{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}^2-(k+1)\beta(B-A)(1-\alpha)^2}{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}^2-\beta(B-A)(1-\alpha)^2} \tag{3.7}$$

The result (3.7) is the best possible for the function f and g defined by (3.6).

Proof. Proceeding as in proof of Theorem 5, we get

$$\rho \le \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}^2-j\beta(B-A)(1-\alpha)^2}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}^2-\beta(B-A)(1-\alpha)^2}$$

$$(j \ge k+1; k \in \mathbb{N}) \tag{3.8}$$

The right hand side of (3.8) being an increasing function of j, by letting $j=k+1$ in (3.8), we obtain (3.7). This completes the proof of Theorem 6.

Theorem 7. *Let the function f defined by (1.4) and the function g defined by (3.1) belong to the class $T^{\lambda}_{n,k}(\alpha,\beta,A,B)$. Then the function h defined by*

$$h(z) = z - \sum_{j=k+1}^{\infty}(a_j^2+b_j^2)z^j$$

belongs to the class $T^{\lambda}_{n,k}(\sigma,\beta,A,B)$, where

$$\sigma = \frac{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}^2 - 2\beta(B-A)(1-\alpha)^2\{(1+\beta B)k+\beta(B-A)\}}{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}^2-2\{\beta(B-A)(1-\alpha)\}^2}$$

This result is sharp for the functions f and g defined by (3.6).

Proof. Suppose $f,g \in T^{\lambda}_{n,k}(\alpha,\beta,A,B)$. Then by Theorem 1, we have

$$\sum_{j=k+1}^{\infty}\left(\frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{\beta(B-A)(1-\alpha)}\right)^2 a_j^2$$

$$\le \left(\sum_{j=k+1}^{\infty}\frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{\beta(B-A)(1-\alpha)}a_j\right)^2 \le 1 \tag{3.9}$$

Similarly, we have

$$\sum_{j=k+1}^{\infty}\left(\frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{\beta(B-A)(1-\alpha)}\right)^2 b_j^2 \le 1 \qquad (3.10)$$

It follows from (3.9) and (3.10) that

$$\sum_{j=k+1}^{\infty}\frac{1}{2}\left(\frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{\beta(B-A)(1-\alpha)}\right)^2 (a_j^2+b_j^2) \le 1$$

Therefore we need to find largest σ such that

$$\frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\sigma)\}}{\beta(B-A)(1-\sigma)}$$
$$\le \frac{1}{2}\left(\frac{j^n(1-\lambda+\lambda_j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{\beta(B-A)(1-\alpha)}\right)^2$$
$$(j \ge k+1; k \in \mathbb{N}),$$

that is,

$$\sigma \le \frac{j^n(1-\lambda+\lambda j)\{(1+\beta B)(j-1)+\beta(B-A)(1-\alpha)\}^2 -2\beta(B-A)(1-\alpha)^2\{(1+\beta B)(j-1)+\beta(B-A)\}}{j^n(1-\lambda+\lambda j)\{(1+\beta B)(j-1)+\beta(B-A)(1-\alpha)\}^2-2\{\beta(B-A)(1-\alpha)\}^2}$$
$$(j \ge k+1; k \in \mathbb{N}) \qquad (3.11)$$

Since the right hand side of (3.11) is an increasing function of j, we have

$$\sigma \le \frac{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}^2 -2\beta(B-A)(1-\alpha)^2\{(1+\beta B)k+\beta(B-A)\}}{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}^2-2\{\beta(B-A)(1-\alpha)\}^2}$$

Thus the Theorem 7 is proved.

4. A Family of Integral Operators

Theorem 8. *Let the function f defined by (1.4) be in the class $T_{n,k}^{\lambda}(\alpha,\beta,A,B)$, and let c be a real number such that $c>-1$. Then the function F defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (c>-1; f\in A_k) \qquad (4.1)$$

belongs to the class $T^{\lambda}_{n,k}(\kappa, \beta, A, B)$*, where*

$$\kappa = \frac{(1+\beta B)\{k+(c+1)\alpha\}+\beta(B-A)(1-\alpha)}{(1+\beta B)\{k+c+1\}+\beta(B-A)(1-\alpha)}$$

This result is sharp for the functions f *defined by (2.8).*

Proof. Form (4.1) we have,

$$F(z) = z - \sum_{j=k+1}^{\infty} \left(\frac{c+1}{c+j}\right) a_j z^j.$$

We need to find largest κ such that

$$\frac{\{(j-1)(1+\beta B)+\beta(B-A)(1-\kappa)\}(c+1)}{(1-\kappa)(c+j)} \leq \frac{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)}{1-\alpha} \quad (j \geq k+1; k \in \mathbb{N})$$

or, equivalently,

$$\kappa \leq \frac{(1+\beta B)\{(j-1)+(c+1)\alpha\}+\beta(B-A)(1-\alpha)}{(1+\beta B)\{c+j\}+\beta(B-A)(1-\alpha)} \quad (j \geq k+1; k \in \mathbb{N}). \quad (4.2)$$

The right hand side of (4.2) being an increasing function of j, we have

$$\kappa \leq \frac{(1+\beta B)\{k+(c+1)\alpha\}+\beta(B-A)(1-\alpha)}{(1+\beta B)\{k+c+1\}+\beta(B-A)(1-\alpha)},$$

which completes the proof of Theorem 8.

Theorem 9. *Let the function* F *given by*

$$F(z) = z - \sum_{j=k+1}^{\infty} d_j z^j \quad (d_j \geq 0; j = k+1, k+2, k+3, \ldots; \quad k \in \mathbb{N})$$

be in the class $T^{\lambda}_{n,k}(\alpha, \beta, A, B)$*, and let* c *be a real number such that* $c > -1$*. Then the function* f *defined by*

$$f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} F(t)dt \quad (c > -1; F \in A_k) \quad (4.3)$$

is univalent in $|z| < R$*, where*

$$R = \inf_{j \geq k+1} \left(\frac{j^{n-1}(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}(c+1)}{\beta(B-A)(1-\alpha)(c+j)} \right)^{1/(j-1)} \tag{4.4}$$

The result (4.4) is sharp.

Proof. We find form (4.3) that,

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{j=k+1}^{\infty} \left(\frac{c+j}{c+1} \right) d_j z^j$$

In order to obtain desired result, it is sufficient to show that

$$|f'(z) - 1| < 1 \text{ whenever } |z| < R,$$

where R is given by (4.4).

Now

$$|f'(z) - 1| \leq \sum_{j=k+1}^{\infty} j \left(\frac{c+j}{c+1} \right) d_j |z|^{j-1}.$$

Thus we have $|f'(z) - 1| < 1$ if

$$\sum_{j=k+1}^{\infty} j \left(\frac{c+j}{c+1} \right) d_j |z|^{j-1} < 1. \tag{4.5}$$

But, by Theorem 1, we know that

$$\sum_{j=k+1}^{\infty} \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{\beta(B-A)(1-\alpha)} d_j \leq 1.$$

Hence (4.5) will be satisfied if

$$\frac{j(c+j)}{c+1}|z|^{j-1} < \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}}{\beta(B-A)(1-\alpha)},$$

That is, if

$$|z| < \left(\frac{j^{n-1}(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}(c+1)}{\beta(B-A)(1-\alpha)(c+j)} \right)^{1/(j-1)}$$

$$(j \geq k+1; k \in \mathbb{N}). \tag{4.6}$$

Therefore the function f given by (4.1) is univalent in $|z| < R$, where R is defined by (4.4). The sharpness is follows if we take

$$f(z) = z - \frac{\beta(B-A)(1-\alpha)(c+j)}{j^{n-1}(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}(c+1)} z^j$$
$$(j \geq k+1; k \in \mathbb{N}). \tag{4.7}$$

5. Applications of Fractional Calculus

In this section we prove distortion theorem for functions belonging to the class $T^{\lambda}_{n,k}(\alpha, \beta, A, B)$, which involve operators of fractional calculus defined as follows.

Definition 1 [5]. *The fractional integral of order μ is defined, for a function f, by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int\limits_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta \quad (\mu > 0), \tag{5.1}$$

where f is analytic function in a simply connected region of the complex plane containing the origin, and the multiplicity of $(z-\zeta)^{1-\mu}$ is removed, by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2 [5]. *The fractional derivative of order μ is defined, for a function f, by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int\limits_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta \quad (0 \leq \mu < 1), \tag{5.2}$$

where f is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3 [5]. *Under the hypothesis of Definition 2, the fractional derivative of order $n + \mu$ is defined, for a function f, by*

$$D_z^{n+\mu} f(z) = \frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{d}z^n}\{D_z^{\mu} f(z)\} \quad (0 \leq \mu < 1; n \in \mathbb{N}_0). \tag{5.3}$$

Theorem 10. *Let the function f defined by (1.4) be in the class $T^{\lambda}_{n,k}(\alpha, \beta, A, B)$.*

Then

$$|D_z^{-\mu}(D^i f(z))|$$
$$\geq \frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1-\frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\,\Gamma(k+2+\mu)}r^k\right)$$
$$(|z|=r<1;\mu>0;i\in\{0,1,\dots,n\}) \tag{5.4}$$

and

$$|D_z^{-\mu}(D^i f(z))|$$
$$\leq \frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1+\frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\,\Gamma(k+2+\mu)}r^k\right)$$
$$(|z|=r<1;\mu>0;i\in\{0,1,\dots,n\}) \tag{5.5}$$

The results (5.4) and (5.5) are sharp.

Proof. We observe that

$$f(z)\in T_{n,k}^{\lambda}(\alpha,\beta,A,B) \quad \Leftrightarrow \quad D^i f(z)\in T_{n-i,k}^{\lambda}(\alpha,\beta,A,B)$$

and that

$$D^i f(z)=z-\sum_{j=k+1}^{\infty} j^i a_j z^j \quad (i\in\mathbb{N}_0)$$

Then from Theorem 1, we have

$$(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\sum_{j=k+1}^{\infty} j^i a_j$$
$$\leq \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta B)+\beta(B-A)(1-\alpha)\}a_j$$
$$\leq \beta(B-A)(1-\alpha),$$

so that

$$\sum_{j=k+1}^{\infty} j^i a_j \leq \frac{\beta(B-A)(1-\alpha)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}}. \tag{5.6}$$

Consider the function $G(z)$ defined by

$$\begin{aligned} G(z) &= \Gamma(2+\mu)z^{-\mu}D_z^{-\mu}(D^i f(z)) \\ &= z - \sum_{j=k+1}^{\infty} \frac{\Gamma(j+1)\Gamma(2+\mu)}{\Gamma(j+1+\mu)} j^i a_j z^j \\ &= z - \sum_{j=k+1}^{\infty} \Phi(j) j^i a_j z^j, \end{aligned}$$

where

$$\Phi(j) = \frac{\Gamma(j+1)\Gamma(2+\mu)}{\Gamma(j+1+\mu)} \quad (j \geq k+1; k \in \mathbb{N}; \mu > 0).$$

Since $\Phi(j)$ is a decreasing function of j, we get

$$0 < \Phi(j) \leq \Phi(k+1) = \frac{\Gamma(k+2)\Gamma(2+\mu)}{\Gamma(k+2+\mu)} \quad (j \geq k+1; k \in \mathbb{N}; \mu > 0). \tag{5.7}$$

Thus by using (5.6) and (5.7), we see that

$$\begin{aligned} |G(z)| &\geq r - \Phi(k+1)r^{k+1} \sum_{j=k+1}^{\infty} j^i a_j \\ &\geq r - \frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\Gamma(k+2+\mu)} r^{k+1} \\ &\quad (|z| = r < 1; \mu > 0; i \in \{0,1,\ldots,n\}) \end{aligned}$$

and

$$\begin{aligned} |G(z)| &\leq r + \Phi(k+1)r^{k+1} \sum_{j=k+1}^{\infty} j^i a_j \\ &\leq r + \frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\Gamma(k+2+\mu)} r^{k+1} \\ &\quad (|z| = r < 1; \mu > 0; i \in \{0,1,\ldots,n\}), \end{aligned}$$

which proves the inequalities (5.4) and (5.5) of Theorem 10.

The inequalities (5.4) and (5.5) are attained for the function $f(z)$ given by

$$D^i f(z) = z - \frac{\beta(B-A)(1-\alpha)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}} z^{k+1} \quad (k \in \mathbb{N}). \tag{5.8}$$

This completes the proof of Theorem 10.

Corollary 2. *Let the function f defined by (1.4) be in the class $T^{\lambda}_{n,k}(\alpha,\beta,A,B)$. Then*

$$|D_z^{-\mu} f(z)| \geq \frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1 - \frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2+\mu)}{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\,\Gamma(k+2+\mu)} r^k\right)$$
$$(|z| = r < 1; \mu > 0) \tag{5.9}$$

and

$$|D_z^{-\mu} f(z)| \leq \frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1 + \frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2+\mu)}{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\,\Gamma(k+2+\mu)} r^k\right)$$
$$(|z| = r < 1; \mu > 0) \tag{5.10}$$

The estimates in (5.9) and (5.10) are sharp for the function f given by (5.8) with $i = 0$.

Theorem 11. *Let the function f defined by (1.4) be in the class $T^{\lambda}_{n,k}(\alpha,\beta,A,B)$.*

Then

$$|D_z^\mu(D^i f(z))| \geq \frac{r^{1-\mu}}{\Gamma(2-\mu)}\left(1-\frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2-\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\Gamma(k+2-\mu)}r^k\right)$$

$$(|z|=r<1; 0\leq\mu<1; i\in\{0,1,\ldots,n-1\}) \tag{5.11}$$

and

$$|D_z^\mu(D^i f(z))| \leq \frac{r^{1-\mu}}{\Gamma(2-\mu)}\left(1+\frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2-\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\Gamma(k+2-\mu)}r^k\right)$$

$$(|z|=r<1; 0\leq\mu<1; i\in\{0,1,\ldots,n-1\}). \tag{5.12}$$

The results (5.11) and (5.12) are sharp.

Proof. Consider the function $H(z)$ defined by

$$\begin{aligned} H(z) &= \Gamma(2-\mu)z^\mu D_z^\mu(D^i f(z)) \\ &= z-\sum_{j=k+1}^{\infty}\frac{\Gamma(j+1)\Gamma(2-\mu)}{\Gamma(j+1-\mu)}j^i a_j z^j \\ &= z-\sum_{j=k+1}^{\infty}\Psi(j)j^i a_j z^j, \end{aligned}$$

where

$$\Psi(j)=\frac{\Gamma(j+1)\Gamma(2-\mu)}{\Gamma(j+1-\mu)} \quad (j\geq k+1; k\in\mathbb{N}; 0\leq\mu<1).$$

Since $\Psi(j)$ is a decreasing function of j, we get

$$0<\Psi(j)\leq\Psi(k+1)=\frac{\Gamma(k+2)\Gamma(2-\mu)}{\Gamma(k+2-\mu)} \quad (j\geq k+1; k\in\mathbb{N}; 0\leq\mu<1). \tag{5.13}$$

Thus by using (5.6) and (5.13), we see that

$$|H(z)| \geq r - \Psi(k+1)r^{k+1} \sum_{j=k+1}^{\infty} j^i a_j$$

$$\geq r - \frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2-\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\Gamma(k+2-\mu)} r^{k+1}$$

$$(|z| = r < 1; 0 \leq \mu < 1; i \in \{0, 1, \ldots, n-1\})$$

and

$$|H(z)| \leq r + \Psi(k+1)r^{k+1} \sum_{j=k+1}^{\infty} j^i a_j$$

$$\leq r + \frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2-\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\Gamma(k+2-\mu)} r^{k+1}$$

$$(|z| = r < 1; 0 \leq \mu < 1; i \in \{0, 1, \ldots, n-1\}),$$

The inequalities (5.11) and (5.11) are attained for the function $f(z)$ given by (5.8). This completes the proof of Theorem 11.

Corollary 3. *Let the function f defined by (1.4) be in the class $T_{n,k}^{\lambda}(\alpha, \beta, A, B)$. Then*

$$|D_z^{\mu} f(z)| \geq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \left(1 - \frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2-\mu)}{(k+1)^{n}(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\}\,\Gamma(k+2-\mu)} r^{k} \right)$$

$$(|z| = r < 1; 0 \leq \mu < 1) \tag{5.14}$$

and

$$|D_z^{\mu} f(z)| \leq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \left(1 + \frac{\beta(B-A)(1-\alpha)\Gamma(2+k)\Gamma(2-\mu)}{(k+1)^n(1+\lambda k)\{(1+\beta B)k+\beta(B-A)(1-\alpha)\} \atop \Gamma(k+2-\mu)} r^k \right)$$

$$(|z| = r < 1; 0 \leq \mu < 1) \qquad (5.15)$$

The estimates in (5.14) and (5.15) are sharp for the function $f(z)$ given by (5.8) with $i = 0$.

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