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SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

Firas Ghanim and Maslina Darus

Abstract. In this paper, we consider some properties such as growth and distortion theorem, coefficient problems, radii of convexity and starlikeness and convex linear combinations for certain subclass of meromorphic p-valent functions with positive coefficients.

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*-corresponding author

1. Introduction

Let A_p denote Let denote the class of functions $f(z)$ normalized by

$$f(z) = z^{-p} + \sum_{n=0}^{\infty} a_n z^n \qquad (p \in N := 1, 2, 3, ...), , \tag{1}$$

which are analytic and p-valent in the punctured unit disk $U = \{z : 0 < |z| < 1\}$.

The functions f in A_p is said to be meromorphically starlike functions of order β if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \beta \qquad (z \in U), p \in N. \tag{2}$$

for some $\beta(0 \leq \beta < p)$. We denote by $S_p^*(\beta)$ the class of all meromorphically starlike functions of order β. Similarly, a function f in A_p is said to be meromorphically convex of order β if and only if

$$Re\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \beta \qquad (z \in U), p \in N. \tag{3}$$

for some $\beta(0 \leq \beta < p)$. We denote by $C_p(\beta)$ the class of all meromorphically convex functions of order β.

The functions of the form (1) was considered by Liu and Srivastava [10], and Raina and Srivastava [13].

Let S_p denote the subclass of A_p consisting of functions of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_n| z^n \tag{4}$$

as studied by Mogra [11] and Liu and Srivastava [10].

For functions f in the class A_p, we define a linear operator D^n by

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = z^{-p} + \sum_{k=0}^{\infty} (k+p+1)\, a_k z^k = \frac{\left(z^{p+1} f(z)\right)'}{z^p},$$

and generally

$$D^n f(z) = D\left(D^{n-1} f(z)\right) = z^{-p} + \sum_{k=0}^{\infty} (k+p+1)^n\, a_k z^k, \tag{5}$$

$(f \in A_p, k \in N)$.
Then it is easily verified that

$$z\left(D^n f(z)\right)' = D^{n+1} f(z) - (p+1)\, D^n f(z), \tag{6}$$

$(f \in A_p, k \in N_0, p \in N)$.

The linear operator D^n was considered, when $p = 1$, by Uralegaddi and Somanatha [18]. More recently, Aouf and Hossen [1], Liu and Srivastava [10], Mograc[11] and Srivastava and Patel [14] presented several results involving the operator D^n for $p \in N$.

Making use of the operator D^n, we say that a function $f \in A_p$ is in the class $S_p^*(k, \beta)$ if it satisfies the following inequality:

$$\left|\frac{z\left(D^k f(z)\right)'}{D^k f(z)} + p\right| \leq \left|\frac{z\left(D^k f(z)\right)'}{D^k f(z)} + 2\beta - p\right|, \qquad (k \in N_0 = N \cup 0)\,. \tag{7}$$

for some $\beta(0 \le \beta < p)$ and for all z in U.

It is easy to check that $S_p^*(0,\beta)$ is the class of meromorphically starlike functions of order β and $S_p^*(0,0)$ gives the meromorphically starlike functions for all $z \in U$. Many important properties and characteristics of various interesting subclasses of the class A_p of meromorphically p-valent functions were investigated extensively by (among others) Aouf and Srivastava [2], Aouf and Hossen [2], Chen and Owa [3], Cho and Owa [4], Joshi and Srivastava [6], Kulkarni, Naik and Srivastava [7], Liu and Srivastava [8],[9], Mogra [11], Owa, Darwish and Aouf [12], Srivastava, Hossen and Aouf [15], Uralegaddi and Somanatha [17], [18], and Yang [19], (see also [16], [5]).
Let us write

$$S_p^*\left[k,\beta\right] = S_p^*\left(k,\beta\right) \cap S_p \tag{8}$$

where S_p is the class of functions of the form (4) that are analytic and p-valent in U.

Next , we obtain the coefficient estimates for the classes $S_p^*\left(k,\beta\right)$ and $S_p^*\left[k,\beta\right]$.

2. Coefficient estimates

Here we provide a sufficient condition for a function, analytic in U to be in $S_p^*\left(k,\beta\right)$.

Theorem 1. *Let the function f be defined by (1). If*

$$\sum_{n=0}^{\infty} \left(n+p+1\right)^k \left(n+p+1+\beta\right)\left|a_n\right| \le p-\beta \qquad (k \in N_0) \tag{9}$$

where $(0 \le \beta < p)$, then $f \in S_p^*(k,\beta)$.

Proof. Suppose that (9) holds true for $0 \le \beta < p$. Consider the expression

$$M\left(f,f'\right) = \left|z\left(D^k f(z)\right)' + pD^k f(z)\right| \le \left|z\left(D^k f(z)\right)' + \left(2\beta - p\right)D^k f(z)\right|.$$

Then for $0 < |z| = r < 1$, we have

$$M\left(f,f'\right) = \left|\sum_{n=0}^{\infty} \left(n+p+1\right)^k \left(n+2p+1\right) a_n z^n\right|$$

$$-\left|\frac{2\left(\beta-p\right)}{z^{p}}+\sum_{n=0}^{\infty}\left(n+p+1\right)^{k}\left(n+1+2\beta\right)a_{n}z^{n}\right|,$$

$$M\left(f,f'\right)\leq\sum_{n=0}^{\infty}\left(n+p+1\right)^{k}\left(n+2p+1\right)\left|a_{n}\right|r^{n}$$

$$-\left(\frac{2\left(p-\beta\right)}{r^{p}}-\sum_{n=0}^{\infty}\left(n+p+1\right)^{k}\left(n+1+2\beta\right)\left|a_{n}\right|r^{n}\right)$$

$$\leq\sum_{n=0}^{\infty}2\left(n+p+1\right)^{k}\left(n+p+1+\beta\right)\left|a_{n}\right|r^{n}-\frac{2\left(p-\beta\right)}{r^{p}}$$

that is,

$$r^{p}M\left(f,f'\right)\leq\sum_{n=0}^{\infty}2\left(n+p+1\right)^{k}\left(n+p+1+\beta\right)\left|a_{n}\right|r^{n+p}-2\left(p-\beta\right)\tag{10}$$

The inequality in (10) holds true for all $r(0\leq r<1)$. Therefore, letting $r\to 1$ in (10), we obtain

$$M\left(f,f'\right)\leq\sum_{n=0}^{\infty}2\left(n+p+1\right)^{k}\left(n+p+1+\beta\right)\left|a_{n}\right|-2\left(p-\beta\right)\leq 0,$$

by the hypothesis (9). Hence it follows that
$\left|\frac{z\left(D^{k}f(z)\right)'}{D^{k}f(z)}+p\right|\leq\left|\frac{z\left(D^{k}f(z)\right)'}{D^{k}f(z)}+2\beta-p\right|$, so that $f\in S_{p}^{*}(k,\beta)$. The result is sharp. Hence the theorem.

Corollary 1. Let $k=\beta=0$ in the Theorem 1, then we have

$$\sum_{n=0}^{\infty}\left(n+p+1\right)\left|a_{n}\right|\leq p.$$

Corollary 2. Let $k=1$ and $\beta=0$ in the Theorem 1, then we have

$$\sum_{n=0}^{\infty}\left(n+p+1\right)^{2}\left|a_{n}\right|\leq p.$$

Next we give a necessary and sufficient condition for a function $f \in S_P$ to be in the class $S_p^*[k,\beta]$.

Theorem 2. Let the function f be defined by (4) and let $f \in S_p$. Then $f \in S_p^*[k,\beta]$ if and only if

$$\sum_{n=p}^{\infty}(n+p+1)^k(n+p+1+\beta)|a_n| \le p-\beta, \tag{11}$$

$(k \in N_0, n = p, p+1, p+2, ..., 0 \le \beta < 1)$.

Proof. In view of Theorem 1, it suffices to show that the 'only if 'part. Assume that $f \in S_p^*[k,\beta]$. Then

$$\left|\frac{\frac{z\left(D^k f(z)\right)'}{D^k f(z)}+p}{\frac{z\left(D^k f(z)\right)'}{D^k f(z)}+2\beta-p}\right| \le$$

$$\left|\frac{\sum\limits_{n=p}^{\infty}(n+p+1)^k(n+2p+1)a_n z^n}{\frac{2(p-\beta)}{z^p}-\sum\limits_{n=p}^{\infty}(n+p+1)^k(n+1+2\beta)a_n z^n}\right| \le 1, \quad (z \in U). \tag{12}$$

Since $Re(z) \le |z|$ for all z, it follows from (12) that

$$\text{Re}\left\{\frac{\sum\limits_{n=p}^{\infty}(n+p+1)^k(n+2p+1)a_n z^n}{\frac{2(p-\beta)}{z^p}-\sum\limits_{n=p}^{\infty}(n+p+1)^k(n+1+2\beta)a_n z^n}\right\} < 1 \qquad (z \in U). \tag{13}$$

We now choose the values z on the real axis so that $\frac{z\left(D^k f(z)\right)'}{D^k f(z)}$ is real. Upon clearing the denominator in (13) and letting $z \to 1$ through real values, we obtain

$$\sum_{n=p}^{\infty}(n+p+1)^k(n+2p+1)a_n$$

$$\le 2(p-\beta)-\sum_{n=p}^{\infty}(n+p+1)^k(n+1+2\beta)a_n, \tag{14}$$

which immediately yields the required condition (9).

Our assertion in Theorem 2 is sharp for functions of the form:

$$f_n(z) = z^{-p} + \frac{p-\beta}{(n+p+1)^k (n+p+1+\beta)} z^n, \tag{15}$$

$(n = p, p+1, p+2, ..., k \in N_0)$.

Corollary 5. Let the function f be defined by (4) and let $f \in S_p$ If $f \in S_p^* [k, \beta]$. Then for fixed n, we have

$$|a_n| \leq \frac{p-\beta}{(n+p+1)^k (n+p+1+\beta)}, \tag{16}$$

$(n = p, p+1, p+2, ..., k \in N_0)$.
The result (16) is sharp for functions f given by (15).

3.Covering theorem

A growth and distortion property for functions in the class $S_p^* [k, \beta]$ is contained in

Theorem 3. *If the function f be defined by (4) is in the class $S_p^* [k, \beta]$ then for $0 < |z| = r < 1$, we have*

$$\left(\frac{(p+m-1)!}{(p-1)!} - \frac{p!\,(p-\beta)}{(p-m)!\,(2p+1)^k (2p+1+\beta)} r^{2p} \right) r^{-(p+m)} \leq |f^m(z)|$$

$$\leq \left(\frac{(p+m-1)!}{(p-1)!} + \frac{p!\,(p-\beta)}{(p-m)!\,(2p+1)^k (2p+1+\beta)} r^{2p} \right) r^{-(p+m)} \tag{17}$$

$(m = 0, 1, 2, 3, ..., p-1)$.
These inequalities are sharp for the function f given by

$$f(z) = z^{-p} + \frac{p-\beta}{(2p+1)^k (2p+1+\beta)} z^p. \tag{18}$$

Proof. Let $f \in S_p^*[k,\beta]$. Then we find from Theorem 2 that

$$\frac{(2p+1)^k(2p+1+\beta)}{p!}\sum_{n=p}^{\infty} n!\,|a_n| \leq$$

$$\sum_{n=p}^{\infty}(n+p+1)^k(n+p+1+\beta)\,|a_n| \leq p-\beta$$

which yields

$$\sum_{n=p}^{\infty} n!\,|a_n| \leq \frac{p!\,(p-\beta)}{(2p+1)^k(2p+1+\beta)}. \tag{19}$$

Now, by differentiating f in (4) m times, we have

$$f^{(m)}(z) = (-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-p-m} + \sum_{n=p}^{\infty}\frac{n!}{(n-m)!}|a_n|\,z^{n-m}. \tag{20}$$

Theorem 3 would readily follow from (19) and (20).

Next, we determine the radii of meromorphically p-valent starlikeness and meromorphically p-valent convexity for functions in the class $S_p^*[k,\beta]$.

4. Radii of Starlikeness and Convexity

Theorem 4. *If the function f be defined by (4) is in the class $S_p^*[k,\beta]$ then f is meromorphically starlike of order $\delta(0 \leq \delta < 1)$ in $|z| < r$, where*

$$r_1 = r_1(k,\beta,\delta) = \inf_{n\geq p}\left\{\frac{(n+p+1)^k(n+p+1+\beta)(p-\delta)}{(n+2p-\gamma)(p-\beta)}\right\}^{\frac{1}{(n+p)}} \tag{21}$$

The result is sharp for the functions f given by (15).

Proof. It sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)}+p\right| \leq p-\delta, \tag{22}$$

for $|z| < r_1$. We have

$$\left|\frac{zf'(z)}{f(z)}+p\right| = \left|\frac{\sum\limits_{n=p}^{\infty}(n+p)\,a_n z^n}{\frac{1}{z^p}+\sum\limits_{n=p}^{\infty}a_n z^n}\right| \leq \frac{\sum\limits_{n=p}^{\infty}(n+p)\,a_n\,|z|^{n+p}}{1-\sum\limits_{n=p}^{\infty}a_n\,|z|^{n+p}}. \tag{23}$$

Hence (23) holds true if

$$\sum_{n=p}^{\infty}(n+p)\,a_n\,|z|^{n+p} \leq (p-\delta)\left(1-\sum_{n=p}^{\infty}a_n\,|z|^{n+p}\right), \tag{24}$$

or

$$\sum_{n=p}^{\infty}\frac{(n+2p-\delta)}{(p-\delta)}a_n\,|z|^{n+p} \leq 1 \tag{25}$$

with the aid of (11) and (25) is true if

$$\frac{(n+2p-\delta)}{(p-\delta)}\,|z|^{n+p} \leq \frac{(n+p+1)^k\,(n+p+1+\beta)}{(p-\beta)}, \qquad (n \geq p)\,. \tag{26}$$

Solving (26) for $|z|$, we obtain

$$|z| \leq \left\{\frac{(n+p+1)^k\,(n+p+1+\beta)\,(p-\delta)}{(n+2p-\gamma)\,(p-\beta)}\right\}^{\frac{1}{n+p}}, \qquad (n \geq p)\,. \tag{27}$$

This completes the proof of Theorem 4.

Theorem 5. *If the function f be defined by (4) is in the class $S_p^*[k,\beta]$ then f is meromorphically convex of order $\delta(0 \leq \delta < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2\,(k,\beta,\delta) = \inf_{n\geq p}\left\{\frac{p\,(n+p+1)^{k-1}\,(n+p+1+\beta)\,(p-\delta)}{(n+2p-\gamma)\,(p-\beta)}\right\}^{\frac{1}{(n+p)}}. \tag{28}$$

The result is sharp for the function f given by (15).

Proof. By using the technique employed in the proof of Theorem 4, we can show that

$$\left|\frac{zf''(z)}{f'(z)}+p+1\right| \leq (1-\delta) \tag{29}$$

for $|z| < r_2$, with the aid of Theorem 2. Thus we have the assertion of Theorem 5.

5.Convex Linear Combinations

Our next result involves convex linear combinations of the functions f of the type (15).

Theorem 6. *Let*

$$f_{p-1}(z) = z^{-p} \tag{30}$$

and

$$f_{n+p-1}(z) = z^{-p} + \frac{(p-\beta)}{(n+p+1)^k\,(n+p+1+\beta)} z^{n+p-1}, \tag{31}$$

$(n \geq p, k \in N_0)$.
Then $f \in S_p^*[k,\beta]$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=p}^{\infty} \lambda_{n+p-1} f_{n+p-1}(z) \tag{32}$$

where $\lambda_{n+p-1} \geq 0$ *and* $\sum\limits_{n=p}^{\infty} \lambda_{n+p-1} = 1$.

Proof: From (32) it is easy to see that

$$f(z) = \sum_{n=p}^{\infty} \lambda_{n+p-1} f_{n+p-1}(z)$$

$$= \frac{1}{z^p} + \sum_{n=p}^{\infty} \frac{(p-\beta)}{(n+p+1)^k\,(n+p+1+\beta)} \lambda_{n+p} z^{n+p}. \tag{33}$$

Since

$$\sum_{n=p}^{\infty} \frac{(n+p+1)^k\,(n+p+1+\beta)}{(p-\beta)} \lambda_{n+p} . \frac{(p-\beta)}{(n+p+1)^k\,(n+p+1+\beta)}$$

$$= \sum_{n=p}^{\infty} \lambda_{n+p} = 1 - \lambda_{p-1} \leq 1,$$

it follows from Theorem 2 that the function $f \in S_p^*[k,\beta]$.

Conversely, let us suppose that $f \in S_p^*[k,\beta]$. Since

$$|a_{n+p}| \leq \frac{(p-\beta)}{(n+p+1)^k\,(n+p+1+\beta)}, \qquad (n \geq p, k \in N_0)\,,$$

Setting

$$\lambda_{n+p} = \frac{(n+p-1)^k\,(n+p+1+\beta)}{(p-\beta)}\,|a_{n+p-1}|\,, \quad (n \geq p, k \in N_0)\,,$$

and $\lambda_{p-1} = 1 - \sum\limits_{n=p}^{\infty} \lambda_{n+p}$,
it follows that $f(z) = \sum\limits_{n=p}^{\infty} \lambda_{n+p-1} f_{n+p-1}(z)$.
This completes the proof of the theorem.

Finally, we prove the following:

Theorem 7. *The class $S_p^*[k,\beta]$ is closed under convex linear combinations.*

Proof: Suppose that the function $f_1(z)$ and $f_2(z)$ defined by

$$f_j\,(z) = z^{-p} + \sum_{n=p}^{\infty} |a_{n,j}| z^n, \qquad (j = 1, 2;\; z \in U)\,, \tag{34}$$

are in the class $S_p^*[k,\beta]$. Setting

$$f\,(z) = \mu f_1\,(z) + (1-\mu)\, f_2\,(z) \qquad (0 \leq \mu \leq 1)\,, \tag{35}$$

we find from (35) that

$$f\,(z) = z^{-p} + \sum_{n=p}^{\infty} \{\mu\,|a_{n,1}| + (1-\mu)\,|a_{n,2}|\} z^n, \qquad (0 \leq \mu \leq 1;\; z \in U)\,. \tag{36}$$

In view of Theorem 2, we have

$$= \mu \sum_{n=p}^{\infty} \left[(n+p-1)^k\,(n+p+1+\beta)\right] |a_{n,1}|$$

$$+ (1-\mu) \sum_{n=p}^{\infty} \left[(n+p-1)^k\,(n+p+1+\beta)\right] |a_{n,2}|$$

$$\leq \mu (p - \beta) + (1 - \mu) (p - \beta) = (p - \beta) .$$

which shows that $f \in S_p^*[k, \beta]$. Hence the theorem.

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F. Ghanim
Faculty of Management
Multimedia University
Cyberjaya63100 Selangor D. Ehsan, Malaysia
email:*firas.ghanim@mmu.edu.my*

M. Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
email:*maslina@ukm.my*