Acta Universitatis Apulensis | No. 25/2011
ISSN: 1582-5329 | pp. 31-40

CERTAIN DIFFERENTIAL SUPERORDINATIONS USING A GENERALIZED SĂLĂGEAN AND RUSCHEWEYH OPERATORS

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Abstract. In the present paper we define a new operator using the generalized Sălăgean and Ruscheweyh operators. Denote by DR_λ^m the Hadamard product of the generalized Sălăgean operator D_λ^m and the Ruscheweyh operator R^m, given by $DR_\lambda^m : \mathcal{A} \to \mathcal{A}$, $DR_\lambda^m f(z) = (D_\lambda^m * R^m) f(z)$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z+a_{n+1}z^{n+1}+\ldots,\ z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. We study some differential superordinations regarding the operator DR_λ^m.

2000 *Mathematics Subject Classification*: 30C45, 30A20, 34A40.

Keywords: Differential superordination, convex function, best subordinant, differential operator.

1. Introduction and definitions

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$ and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

If f and g are analytic functions in U, we say that f is superordinate to g, written $g \prec f$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $g(z) = f(w(z))$ for all $z \in U$. If f is univalent, then $g \prec f$ if and only if $f(0) = g(0)$ and $g(U) \subseteq f(U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi(p(z), zp'(z); z)$ are univalent in U and satisfies the (first-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \qquad \text{for } \ z \in U, \tag{1}$$

then p is called a solution of the differential superordination. The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all p satisfying (1). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant of (1). The best subordinant is unique up to a rotation of U.

Definition 1 *(Al Oboudi [4]) For $f \in \mathcal{A}$, $\lambda \geq 0$ and $m \in \mathbb{N}$, the operator D_λ^m is defined by $D_\lambda^m : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \\ &... \\ D_\lambda^m f(z) &= (1-\lambda) D_\lambda^{m-1} f(z) + \lambda z \left(D_\lambda^m f(z)\right)' = D_\lambda \left(D_\lambda^{m-1} f(z)\right), \quad for \quad z \in U. \end{aligned}$$

Remark 1 If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $D_\lambda^m f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m a_j z^j$, for $z \in U$.

Remark 2 For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [7].

Definition 2 *(Ruscheweyh [6]) For $f \in \mathcal{A}$, $m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &... \\ (m+1) R^{m+1} f(z) &= z \left(R^m f(z)\right)' + m R^m f(z), \quad z \in U. \end{aligned}$$

Remark 3 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^m f(z) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m a_j z^j$, $z \in U$.

Definition 3 *([5]) We denote by Q the set of functions that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.*

We will use the following lemmas.

Lemma 1 *(Miller and Mocanu [5]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \backslash \{0\}$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and*

$$h(z) \prec p(z) + \frac{1}{\gamma} z p'(z), \quad for \quad z \in U,$$

then

$$q(z) \prec p(z), \quad for \quad z \in U,$$

where $q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$, *for* $z \in U$. *The function* q *is convex and is the best subordinant.*

Lemma 2 *(Miller and Mocanu [5]) Let* q *be a convex function in* U *and let* $h(z) = q(z) + \frac{1}{\gamma} z q'(z)$, *for* $z \in U$, *where* $Re\ \gamma \geq 0$.

If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ *is univalent in* U *and*

$$q(z) + \frac{1}{\gamma} z q'(z) \prec p(z) + \frac{1}{\gamma} z p'(z), \quad \text{for } z \in U,$$

then

$$q(z) \prec p(z), \quad \text{for } z \in U,$$

where $q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$, *for* $z \in U$. *The function* q *is the best subordinant.*

2. Main results

Definition 4 *([2]) Let* $\lambda \geq 0$ *and* $m \in \mathbb{N}$. *Denote by* DR_λ^m *the operator given by the Hadamard product (the convolution product) of the generalized Sălăgean operator* D_λ^m *and the Ruscheweyh operator* R^m, $DR_\lambda^m : \mathcal{A} \to \mathcal{A}$,

$$DR_\lambda^m f(z) = (D_\lambda^m * R^m) f(z).$$

Remark 4 If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then
$DR_\lambda^m f(z) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m [1 + (j-1)\lambda]^m a_j^2 z^j$, for $z \in U$.

Remark 5 For $\lambda = 1$ we obtain the Hadamard product SR^n [1] of the Sălăgean operator S^n and Ruscheweyh operator R^n.

Theorem 1 *Let* h *be a convex function,* $h(0) = 1$. *Let* $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ *and suppose that* $\frac{m+1}{(m\lambda+1)z} DR_\lambda^{m+1} f(z) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_\lambda^m f(z)$ *is univalent and* $(DR_\lambda^m f(z))' \in \mathcal{H}[1,1] \cap Q$. *If*

$$h(z) \prec \frac{m+1}{(m\lambda+1)z} DR_\lambda^{m+1} f(z) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_\lambda^m f(z), \quad \text{for } z \in U, \tag{2}$$

then

$$q(z) \prec (DR_\lambda^m f(z))', \quad \text{for } z \in U,$$

where $q(z) = \frac{m+\frac{1}{\lambda}}{z^{m+\frac{1}{\lambda}}} \int_0^z h(t) t^{m-1+\frac{1}{\lambda}} dt$. *The function* q *is convex and it is the best subordinant.*

Proof. With notation $p(z) = (DR_\lambda^m f(z))' = 1 + \sum_{j=2}^{\infty} C_{m+j-1}^m [1 + (j-1)\lambda]^m \cdot$
$j a_j^2 z^{j-1}$ and $p(0) = 1$, we obtain for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$,

$p(z)+zp'(z)=\frac{m+1}{\lambda z}DR_\lambda^{m+1}f(z)-\left(m-1+\frac{1}{\lambda}\right)\left(DR_\lambda^m f(z)\right)'-\frac{m(1-\lambda)}{\lambda z}DR_\lambda^m f(z)$ and $p(z)+\frac{\lambda}{m\lambda+1}zp'(z)=\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z)$.
Evidently $p\in\mathcal{H}[1,1]$.

Then (2) becomes

$$h(z)\prec p(z)+\frac{\lambda}{m\lambda+1}zp'(z),\quad \text{for}\ \ z\in U.$$

By using Lemma 1 for $\gamma=m+\frac{1}{\lambda}$ and $n=1$, we have

$$q(z)\prec p(z),\quad \text{for}\ \ z\in U,\quad \text{i.e.}\quad q(z)\prec\left(DR_\lambda^m f(z)\right)',\quad \text{for}\ \ z\in U,$$

where $q(z)=\frac{m+\frac{1}{\lambda}}{z^{m+\frac{1}{\lambda}}}\int_0^z h(t)\,t^{m-1+\frac{1}{\lambda}}dt$. The function q is convex and it is the best subordinant.

Corollary 1 *([3]) Let h be a convex function, $h(0)=1$. Let $n\in\mathbb{N}$, $f\in\mathcal{A}$ and suppose that $\frac{1}{z}SR^{n+1}f(z)+\frac{n}{n+1}z\left(SR^n f(z)\right)''$ is univalent and $\left(SR^n f(z)\right)'\in\mathcal{H}[1,1]\cap Q$. If*

$$h(z)\prec\frac{1}{z}SR^{n+1}f(z)+\frac{n}{n+1}z\left(SR^n f(z)\right)'',\quad \textit{for}\ \ z\in U,\tag{3}$$

then

$$q(z)\prec\left(SR^n f(z)\right)',\quad \textit{for}\ \ z\in U,$$

where $q(z)=\frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the best subordinant.

Theorem 2 *Let q be convex in U and let h be defined by $h(z)=q(z)+\frac{\lambda}{m\lambda+1}zq'(z)$, $\lambda\geq 0$, $m\in\mathbb{N}$. If $f\in\mathcal{A}$, suppose that $\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z)-\frac{m(1-\lambda)}{(m\lambda+1)z}\cdot$ $DR_\lambda^m f(z)$ is univalent, $\left(DR_\lambda^m f(z)\right)'\in\mathcal{H}[1,1]\cap Q$ and satisfies the differential superordination*

$$h(z)=q(z)+\frac{\lambda}{m\lambda+1}zq'(z)\prec\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z),\tag{4}$$

for $z\in U$, then

$$q(z)\prec\left(DR_\lambda^m f(z)\right)',\quad \textit{for}\ \ z\in U,$$

where $q(z)=\frac{m+\frac{1}{\lambda}}{z^{m+\frac{1}{\lambda}}}\int_0^z h(t)\,t^{m-1+\frac{1}{\lambda}}dt$. The function q is the best subordinant.

Proof. Let $p(z)=\left(DR_\lambda^m f(z)\right)'=1+\sum_{j=2}^\infty C_{m+j-1}^m\left[1+(j-1)\lambda\right]^m ja_j^2z^{j-1}$.

Differentiating, we obtain $p(z)+zp'(z)=\frac{m+1}{\lambda z}DR_\lambda^{m+1}f(z)-\left(m-1+\frac{1}{\lambda}\right)\cdot$ $\left(DR_\lambda^m f(z)\right)'-\frac{m(1-\lambda)}{\lambda z}DR_\lambda^n f(z)$ and $p(z)+\frac{\lambda}{m\lambda+1}zp'(z)=\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z)-$

$\frac{m(1-\lambda)}{(m\lambda+1)z} DR_{\lambda}^{m} f(z)$, for $z \in U$ and (4) becomes

$$q(z) + \frac{\lambda}{m\lambda+1} zq'(z) \prec p(z) + \frac{\lambda}{m\lambda+1} zp'(z), \quad \text{for } z \in U.$$

Using Lemma 2 for $\gamma = m + \frac{1}{\lambda}$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for } z \in U, \quad \text{i.e. } q(z) = \frac{m+\frac{1}{\lambda}}{z^{m+\frac{1}{\lambda}}} \int_0^z h(t)\, t^{m-1+\frac{1}{\lambda}} dt \prec \left(DR_{\lambda}^{m} f(z)\right)',$$

for $z \in U$, and q is the best subordinant.

Corollary 2 *([3]) Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $n \in \mathbb{N}$, $f \in \mathcal{A}$, suppose that $\frac{1}{z} SR^{n+1} f(z) + \frac{n}{n+1} z \left(SR^{n} f(z)\right)''$ is univalent, $\left(SR^{n} f(z)\right)' \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \frac{1}{z} SR^{n+1} f(z) + \frac{n}{n+1} z \left(SR^{n} f(z)\right)'', \quad \text{for } z \in U, \tag{5}$$

then

$$q(z) \prec \left(SR^{n} f(z)\right)', \quad \text{for } z \in U,$$

where $q(z) = \frac{1}{z} \int_0^z h(t) dt$. The function q is the best subordinant.

Theorem 3 *Let h be a convex function, $h(0) = 1$. Let $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\left(DR_{\lambda}^{m} f(z)\right)'$ is univalent and $\frac{DR_{\lambda}^{m} f(z)}{z} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \left(DR_{\lambda}^{m} f(z)\right)', \quad \text{for } z \in U, \tag{6}$$

then

$$q(z) \prec \frac{DR_{\lambda}^{m} f(z)}{z}, \quad \text{for } z \in U,$$

where $q(z) = \frac{1}{z} \int_0^z h(t) dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z) = \frac{DR_{\lambda}^{m} f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} C_{m+j-1}^{m} [1+(j-1)\lambda]^{m} a_j^2 z^j}{z} =$

$1 + \sum_{j=2}^{\infty} C_{m+j-1}^{m} [1+(j-1)\lambda]^{m} a_j^2 z^{j-1}$. Evidently $p \in \mathcal{H}[1,1]$.

We have $p(z) + zp'(z) = \left(DR_{\lambda}^{m} f(z)\right)'$, for $z \in U$.

Then (6) becomes

$$h(z) \prec p(z) + zp'(z), \quad \text{for } z \in U.$$

By using Lemma 1 for $\gamma = 1$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for } z \in U, \quad \text{i.e. } q(z) \prec \frac{DR_{\lambda}^{m} f(z)}{z}, \quad \text{for } z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the best subordinant.

Corollary 3 *([3]) Let h be a convex function, $h(0) = 1$. Let $n \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $(SR^n f(z))'$ is univalent and $\frac{SR^n f(z)}{z} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec (SR^n f(z))', \quad \text{for } z \in U, \tag{7}$$

then

$$q(z) \prec \frac{SR^n f(z)}{z}, \quad \text{for } z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the best subordinant.

Theorem 4 *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$, suppose that $(DR_\lambda^m f(z))'$ is univalent, $\frac{DR_\lambda^m f(z)}{z} \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec (DR_\lambda^m f(z))', \quad \text{for } z \in U, \tag{8}$$

then

$$q(z) \prec \frac{DR_\lambda^m f(z)}{z}, \quad \text{for } z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is the best subordinant.

Proof. Let $p(z) = \frac{DR_\lambda^m f(z)}{z} = \frac{z+\sum_{j=2}^{\infty} C_{m+j-1}^m [1+(j-1)\lambda]^m a_j^2 z^j}{z} =$

$1 + \sum_{j=2}^{\infty} C_{m+j-1}^m [1+(j-1)\lambda]^m a_j^2 z^{j-1}$. Evidently $p \in \mathcal{H}[1,1]$.

Differentiating, we obtain $p(z) + zp'(z) = (DR_\lambda^m f(z))'$, for $z \in U$ and (8) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad \text{for } z \in U.$$

Using Lemma 2 for $\gamma = 1$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for } z \in U, \quad \text{i.e.} \quad q(z) = \frac{1}{z}\int_0^z h(t)dt \prec \frac{DR_\lambda^m f(z)}{z}, \quad \text{for } z \in U,$$

and q is the best subordinant.

Corollary 4 *([3]) Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $n \in \mathbb{N}$, $f \in \mathcal{A}$, suppose that $(SR^n f(z))'$ is univalent, $\frac{SR^n f(z)}{z} \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec (SR^n f(z))', \quad \text{for } z \in U, \tag{9}$$

then

$$q(z) \prec \frac{SR^n f(z)}{z}, \quad \text{for } z \in U,$$

where $q(z)=\frac{1}{z}\int_0^z h(t)dt$. *The function* q *is the best subordinant.*

Theorem 5 *Let* $h(z)=\frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U, *where* $0\leq\beta<1$. *Let* $\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}$ *and suppose that* $\left(DR_\lambda^m f(z)\right)'$ *is univalent and* $\frac{DR_\lambda^m f(z)}{z}\in\mathcal{H}[1,1]\cap Q$. *If*

$$h(z)\prec\left(DR_\lambda^m f(z)\right)',\quad \text{for } z\in U, \tag{10}$$

then

$$q(z)\prec\frac{DR_\lambda^m f(z)}{z},\quad \text{for } z\in U,$$

where q *is given by* $q(z)=2\beta-1+2(1-\beta)\frac{\ln(1+z)}{z}$, *for* $z\in U$. *The function* q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem and considering $p(z)=\frac{DR_\lambda^m f(z)}{z}$, the differential superordination (10) becomes

$$h(z)=\frac{1+(2\beta-1)z}{1+z}\prec p(z)+zp'(z),\quad \text{for } z\in U.$$

By using Lemma 1 for $\gamma=1$ and $n=1$, we have $q(z)\prec p(z)$, i.e.,

$$q(z)=\frac{1}{z}\int_0^z h(t)dt=\frac{1}{z}\int_0^z\frac{1+(2\beta-1)t}{1+t}dt=2\beta-1+2(1-\beta)\frac{1}{z}\ln(z+1)\prec\frac{DR_\lambda^m f(z)}{z},$$

for $z\in U$.
The function q is convex and it is the best subordinant.

Theorem 6 *Let* h *be a convex function,* $h(0)=1$. *Let* $\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}$ *and suppose that* $\left(\frac{zDR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}\right)'$ *is univalent and* $\frac{DR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}\in\mathcal{H}[1,1]\cap Q$. *If*

$$h(z)\prec\left(\frac{zDR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}\right)',\quad \text{for } z\in U, \tag{11}$$

then

$$q(z)\prec\frac{DR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)},\quad \text{for } z\in U,$$

where $q(z)=\frac{1}{z}\int_0^z h(t)dt$. *The function* q *is convex and it is the best subordinant.*

Proof. Consider $p(z)=\frac{DR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}=\frac{z+\sum_{j=2}^\infty C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2z^j}{z+\sum_{j=2}^\infty C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2z^j}=$ $\frac{1+\sum_{j=2}^\infty C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2z^{j-1}}{1+\sum_{j=2}^\infty C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2z^{j-1}}$. Evidently $p\in\mathcal{H}[1,1]$.

We have $p'(z) = \frac{\left(DR_\lambda^{m+1}f(z)\right)'}{DR_\lambda^m f(z)} - p(z) \cdot \frac{\left(DR_\lambda^m f(z)\right)'}{DR_\lambda^m f(z)}$.

Then $p(z) + zp'(z) = \left(\frac{zDR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}\right)'$.

Then (11) becomes

$$h(z) \prec p(z) + zp'(z), \quad \text{for} \ \ z \in U.$$

By using Lemma 1 for $\gamma = 1$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for} \ \ z \in U, \quad \text{i.e.} \quad q(z) \prec \frac{DR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}, \quad \text{for} \ \ z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the best subordinant.

Corollary 5 *([3]) Let h be a convex function, $h(0) = 1$. Let $n \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\left(\frac{zSR^{n+1}f(z)}{SR^n f(z)}\right)'$ is univalent and $\frac{SR^{n+1}f(z)}{SR^n f(z)} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \left(\frac{zSR^{n+1}f(z)}{SR^n f(z)}\right)', \quad \text{for} \ \ z \in U, \tag{12}$$

then

$$q(z) \prec \frac{SR^{n+1}f(z)}{SR^n f(z)}, \quad \text{for} \ \ z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the best subordinant.

Theorem 7 *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$, suppose that $\left(\frac{zDR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}\right)'$ is univalent, $\frac{DR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)} \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zDR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}\right)', \quad \text{for} \ \ z \in U, \tag{13}$$

then

$$q(z) \prec \frac{DR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)}, \quad \text{for} \ \ z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is the best subordinant.

Proof. Let $p(z) = \frac{DR_\lambda^{m+1}f(z)}{DR_\lambda^m f(z)} = \frac{z+\sum_{j=2}^\infty C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2 z^j}{z+\sum_{j=2}^\infty C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2 z^j} =$

$\frac{1+\sum_{j=2}^\infty C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2 z^{j-1}}{1+\sum_{j=2}^\infty C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2 z^{j-1}}$. Evidently $p \in \mathcal{H}[1,1]$.

Differentiating, we obtain $p(z) + zp'(z) = \left(\frac{zDR_\lambda^{m+1} f(z)}{DR_\lambda^m f(z)}\right)'$, for $z \in U$ and (13) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad \text{for } z \in U.$$

Using Lemma 2 for $\gamma = 1$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for } z \in U, \quad \text{i.e. } q(z) = \frac{1}{z}\int_0^z h(t)dt \prec \frac{DR_\lambda^{m+1} f(z)}{DR_\lambda^m f(z)}, \quad \text{for } z \in U,$$

and q is the best subordinant.

Corollary 6 *([3]) Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $n \in \mathbb{N}$, $f \in \mathcal{A}$, suppose that $\left(\frac{zSR^{n+1} f(z)}{SR^n f(z)}\right)'$ is univalent, $\frac{SR^{n+1} f(z)}{SR^n f(z)} \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zSR^{n+1} f(z)}{SR^n f(z)}\right)', \quad \text{for } z \in U, \tag{14}$$

then

$$q(z) \prec \frac{SR^{n+1} f(z)}{SR^n f(z)}, \quad \text{for } z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is the best subordinant.

Theorem 8 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \le \beta < 1$. Let $\lambda \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\left(\frac{zDR_\lambda^{m+1} f(z)}{DR_\lambda^m f(z)}\right)'$ is univalent, $\frac{DR_\lambda^{m+1} f(z)}{DR_\lambda^m f(z)} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \left(\frac{zDR_\lambda^{m+1} f(z)}{DR_\lambda^m f(z)}\right)', \quad \text{for } z \in U, \tag{15}$$

then

$$q(z) \prec \frac{DR_\lambda^{m+1} f(z)}{DR_\lambda^m f(z)}, \quad \text{for } z \in U,$$

where q is given by $q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}$, for $z \in U$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem and considering $p(z) = \frac{DR_\lambda^{m+1} f(z)}{DR_\lambda^m f(z)}$, the differential superordination (15) becomes

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z} \prec p(z) + zp'(z), \quad \text{for } z \in U.$$

By using Lemma 1 for $\gamma = 1$ and $n = 1$, we have $q(z) \prec p(z)$, i.e.,

$$q(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = 2\beta-1+2(1-\beta)\frac{1}{z}\ln(z+1) \prec \frac{DR_\lambda^{m+1} f(z)}{DR_\lambda^m f(z)},$$

for $z \in U$.
The function q is convex and it is the best subordinant.

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