Acta Universitatis Apulensis No. 26/2011
ISSN: 1582-5329 pp. 69-74

SOME PROPERTIES OF AN INTEGRAL OPERATOR DEFINED BY BESSEL FUNCTIONS

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Abstract. In this paper we will study the integral operator involving Bessel functions of the first kind and of order v. We will investigate the integral operator for the classes of starlike and convex functions in the open unit disk.

Key Words: Bessel function, Starlike function, Convex function.

2000 *Mathematics Subject Classification*: 30C45, 30C50.

1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic in the open unit disc $E = \{z : |z| < 1\}$ and S denote the class of all functions in A which are univalent in E. Also let $C(\alpha)$ and $S^*(\alpha)$ be the subclasses of S consisting of all functions which are respectively convex and starlike of order α $(0 \le \alpha < 1)$. The Bessel functions of the first kind of order v is defined by

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+v}}{n!\, \Gamma(n+v+1)},\ v \in \mathbb{R}, \tag{1.2}$$

where $\Gamma(.)$ denotes the gamma function. Sazász and Kupán [10] have studied the univalence of normalized Bessel functions

$$g_v(z) = 2^v \Gamma(v+1) z^{1-v/2} J_v\left(z^{1/2}\right) = z + \sum_{n=1}^{\infty} \frac{4^{-n} (-1)^n z^{n+1}}{n!\,(v+1)\ldots(v+n)}. \tag{1.3}$$

Later, Selinger [9], Sazász and Kupán [10], Baricz and Ponnusamy [1] obtained the conditions for starlikeness of (1.2) by using different techniques.

Recently, Baricz and Frasin [2] have investigated the univalence of the integral operator given by

$$F(z) = F_{v_1,,\dots v_n,\alpha_1,\dots\alpha_n,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{g_{v_i}(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right\}^{1/\beta}. \tag{1.4}$$

For the integral operator of the form (1.4) which involve analytic functions of the form (1.1), see [3, 4, 5, 8].

In the present paper, we will find the order of starlikeness and convexity for the above integral defined by (1.4) using the result given by Sazász and Kupán [10].

2. Preliminary Lemmas

In order to derive our main results, we need the following lemmas.

Lemma 2.1 [10] *If $v > \frac{\sqrt{3}}{2} - 1$, then the function g_v defined by (1.3) is starlike of order $\frac{1}{2}$ in E.*

Lemma 2.2 [7] *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\psi(u, v)$ be a complex valued function satisfying the conditions:*

(i) $\psi(u, v)$ is continuous in a domain $D \subset C^2$,

(ii) $(1, 0) \in D$ and $\mathrm{Re}\psi(1, 0) > 0$,

(iii) $\mathrm{Re}\psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}\left(1 + u_2^2\right)$.

If $h(z) = 1 + c_1 z + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\mathrm{Re}\psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\mathrm{Re}h(z) > 0$ in E.

3. Main Results

Theorem 3.1. *Let $g_{v_i}(z) \in S^*\left(\frac{1}{2}\right)$, for all $v_i > \frac{\sqrt{3}}{2} - 1$, $i = 1, 2, \dots n$. Then $F(z) \in S^*(\delta)$ with $\alpha_1, \dots \alpha_n, \beta$ are positive real numbers such that*

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \leq 2\beta,$$

where

$$\delta = \frac{-\left(\sum_{i=1}^{n} \frac{1}{\alpha_i} - 2\beta + 1\right) + \sqrt{\left(\sum_{i=1}^{n} \frac{1}{\alpha_i} - 2\beta + 1\right)^2 + 8\beta}}{4\beta}, \quad 0 \leq \delta < 1. \tag{3.1}$$

Proof. Let

$$\frac{zF'(z)}{F(z)} = (1-\delta)\,p(z) + \delta. \tag{3.2}$$

Differentiation of (1.4) and by using(3.2), we have

$$\frac{z^{\beta}\prod_{i=1}^{n}\left(\frac{g_{v_i}(z)}{z}\right)^{\frac{1}{\alpha_i}}}{\left(F(z)\right)^{\beta}} = (1-\delta)\,p(z) + \delta. \tag{3.3}$$

Differentiating (3.3) logarithmically, we obtain

$$\sum_{i=1}^{n}\frac{1}{\alpha_i}\frac{zg'_{v_i}(z)}{g_{v_i}(z)} = \beta\,(1-\delta)\,p(z) + \frac{(1-\delta)\,zp'(z)}{(1-\delta)\,p(z)+\delta} + \sum_{i=1}^{n}\frac{1}{\alpha_i} - \beta\,(1-\delta)\,.$$

Since $g_{v_i}(z) \in S^*\left(\frac{1}{2}\right)$, for all $v_i > \frac{\sqrt{3}}{2} - 1$, $i = 1, 2, \ldots n$, by Lemma 2.1, it follows that

$$\sum_{i=1}^{n}\frac{1}{\alpha_i}\operatorname{Re}\frac{zg'_{v_i}(z)}{g_{v_i}(z)} = \operatorname{Re}\left\{\beta\,(1-\delta)\,p(z) + \frac{(1-\delta)\,zp'(z)}{(1-\delta)\,p(z)+\delta} + \sum_{i=1}^{n}\frac{1}{\alpha_i} - \beta\,(1-\delta)\right\}. \tag{3.4}$$

We now form the functional $\psi(u,v)$ by choosing $u = p(z)$, $v = zp(z)$ in (3.4) and note that the first two conditions of Lemma 2.2 are clearly satisfied. We check condition (iii) as follows.

$$\psi(u,v) = \beta\,(1-\delta)\,u + \frac{(1-\delta)\,v}{(1-\delta)\,u+\delta} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i} - \beta\,(1-\delta)\,.$$

Now

$$\psi(iu_2,v_1) = \beta\,(1-\delta)\,iu_2 + \frac{(1-\delta)\,v_1}{(1-\delta)\,iu_2+\delta} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i} - \beta\,(1-\delta)\,.$$

Taking real part of $\psi(iu_2,v_1)$, we have

$$\operatorname{Re}\psi(iu_2,v_1) = \frac{\delta\,(1-\delta)\,v_1}{(1-\delta)^2\,u_2^2+\delta^2} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i} - \beta\,(1-\delta)\,.$$

Applying $v_1 \le -\frac{1}{2}\left(1+u_2^2\right)$ and after a little simplification, we have

$$\operatorname{Re}\psi(iu_2,v_1) \le \frac{A+Bu_2^2}{2C}, \tag{3.5}$$

where

$$\begin{aligned} A &= \delta^2\left(\sum_{i=1}^{n}\frac{1}{\alpha_i}-2\beta\left(1-\delta\right)\right)-\delta\left(1-\delta\right), \\ B &= \left(1-\delta\right)^2\left(\sum_{i=1}^{n}\frac{1}{\alpha_i}-2\beta\left(1-\delta\right)\right)-\delta\left(1-\delta\right), \\ C &= \left(1-\delta\right)^2 u_2^2+\delta^2. \end{aligned}$$

The right hand side of (3.5) is negative if $A \leq 0$ and $B \leq 0$. From $A_1 \leq 0$, we have the value of δ given by (3.1) and from $B \leq 0$, we have $0 \leq \delta < 1$. Since all the conditions of Lemma 2.2 are satisfied, it follows that $p(z) \in P$ in E and consequently $F(z) \in S^*(\delta)$.

Corollary 3.2. *Let* $g_{v_i}(z) \in S^*\left(\frac{1}{2}\right)$, *for all* $v_i > \frac{\sqrt{3}}{2}-1$, $i = 1, 2, \ldots n$, *and let* $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$. *Then* $F(z) \in S^*(\delta_1)$, α, β *be positive real numbers such that* $n \leq 2\alpha\beta$, *where*

$$\delta_1 = \frac{-\left(\frac{n}{\alpha}-2\beta+1\right)+\sqrt{\left(\frac{n}{\alpha}-2\beta+1\right)^2+8\beta}}{4\beta}, \quad 0 \leq \delta_1 < 1. \tag{3.6}$$

Corollary 3.3. *For* $n = 1$ *in Theorem 3.1,* $F_{v,\alpha,\beta}(z) \in S^*(\delta_2)$, α, β *be positive real numbers such that* $1 \leq 2\alpha\beta$, *where*

$$\delta_2 = \frac{-\left(\frac{1}{\alpha}-2\beta+1\right)+\sqrt{\left(\frac{1}{\alpha}-2\beta+1\right)^2+8\beta}}{4\beta}, \quad 0 \leq \delta_2 < 1. \tag{3.7}$$

Corollary 3.4. *For* $n = 1$, $\beta = 1$ *in Theorem 3.1,* $F_{v,\alpha}(z) \in S^*(\delta_3)$, α *be positive real numbers such that* $1 \leq 2\alpha$, *where*

$$\delta_3 = \frac{-\left(\frac{1}{\alpha}-1\right)+\sqrt{\left(\frac{1}{\alpha}-1\right)^2+8}}{4}, \quad 0 \leq \delta_3 < 1. \tag{3.7}$$

Theorem 3.5. *Let* $g_{v_i}(z) \in S^*\left(\frac{1}{2}\right)$, *for all* $v_i > \frac{\sqrt{3}}{2}-1$, $i = 1, 2, \ldots n$. *Then* $F(z) \in C(\eta)$ *with* $\alpha_1, \alpha_2, \ldots \alpha_n$ *are positive real numbers such that* $0 \leq \eta < 1$, *where*

$$\eta = 1-\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i}. \tag{3.8}$$

Proof. Differentiating (1.4) for $\beta = 1$, we have

$$F'(z) = \prod_{i=1}^{n} \left(\frac{g_{v_i}(z)}{z} \right)^{\frac{1}{\alpha_i}}. \tag{3.9}$$

Now differentiating (3.9) logarithmically, we obtain

$$\frac{zF''(z)}{F'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)} - 1 \right).$$

This implies that

$$1 + \frac{zF''(z)}{F'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)} \right) + \left(1 - \sum_{i=1}^{n} \frac{1}{\alpha_i} \right).$$

Since $g_{v_i}(z) \in S^*\left(\frac{1}{2}\right)$, for all $v_i > \frac{\sqrt{3}}{2} - 1$, $i = 1, 2, \ldots n$, by Lemma 2.1, it follows that

$$\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)} \right) > \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\alpha_i} + \left(1 - \sum_{i=1}^{n} \frac{1}{\alpha_i} \right), \tag{3.10}$$

that is

$$\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)} \right) > \left(1 - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\alpha_i} \right). \tag{3.11}$$

This shows that $F(z) \in C(\eta)$, where the value of η is given by (3.8).

Corollary 3.6. For $n = 1$ in the above theorem, then $F_{v,\alpha}(z) \in C(\eta_1)$, where

$$\eta_1 = 1 - \frac{1}{2\alpha}.$$

Acknowledgment : The principal author would like to express gratitude to Professor Dr. Ihsan Ali, Vice Chancellor Abdul Wali Khan University Mardan for providing excellent research facilities.

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