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INCLUSION RELATIONSHIPS FOR CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH A FAMILY OF MULTIPLIER TRANSFORMATIONS AND HYPEGEOMETRIC FUNCTIONS

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Abstract. The purpose of the present article is to introduce several new subclasses of meromorphic functions defined by using the multiplier transformation and hypergeometric function and investigate various inclusion relationships for these subclasses. Some interesting applications involving a certain class of hypergeometric functions are also considered.

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1. Introduction

Let M denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \ , \tag{1.1}$$

which are analytic in the punctured open unit disc $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U/\{0\}$. Given two parameters η and β $(0 \leq \eta, \beta < 1)$, we denote by $MS(\eta)$, $MK(\eta)$ and $MC(\eta, \beta)$ the subclasses of M consisting of all meromorphic functions which are, respectively, starlike of order η in U, convex of order η in U, and close-to-convex of order β and type η in U, see, for details, refs. [7, 10, 12, 17].

Let N^* be the class of all functions φ which are analytic and univalent in U and for which $\varphi(U)$ is convex with

$$\varphi(0) = 1 \qquad \text{and} \qquad \mathrm{Re}\{\varphi(z)\} > 0 \qquad (z \in U) \ .$$

For functions f and g analytic in $U = U^* \cup \{0\}$, we say that f is subordinate to g, and write

$$f \prec g \text{ in } U \text{ or } f(z) \prec g(z) \qquad (z \in U) ,$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$. It is known that

$$f(z) \prec g(z) \ \ (z \in U) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U) .$$

Furthermore, if the function g is univalent in U, then [see, e.g., 12, p. 4]

$$f(z) \prec g(z) \ \ (z \in U) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U) .$$

Making use of the principle of subordination between analytic functions, we introduce the subclasses $MS(\eta;\varphi)$, $MK(\eta;\varphi)$, and $MC(\eta,\beta;\varphi,\psi)$ of the class M for $0 \leq \eta,\ \beta < 1$ and for $\varphi, \psi \in N^*$, which are defined by

$$MS(\eta;\varphi) := \left\{ f : f \in M \text{ and } \frac{1}{1-\eta}\left(-\frac{zf'(z)}{f(z)} - \eta\right) \prec \varphi(z), \quad (z \in U) \right\} ,$$

$$MK(\eta;\varphi) := \left\{ f : f \in M \text{ and } \frac{1}{1-\eta}\left(-\left[1 + \frac{zf''(z)}{f'(z)}\right] - \eta\right) \prec \varphi(z), \quad (z \in U) \right\} ,$$

and

$$\begin{aligned} MC(\eta,\beta;\varphi,\psi) \ \ : \ \ = & \Big\{ f : f \in M \text{ and } \exists\ g \in MS(\eta;\varphi) : \\ & \frac{1}{1-\beta}\left(-\frac{zf'(z)}{g(z)} - \beta\right) \prec \psi(z), \quad (z \in U) \Big\} , \end{aligned}$$

respectively, we note the classes mentioned above contain various subclasses of meromorphic functions for special choices for the functions φ and ψ (as well as for special choices for the parameters η and β) involved in these definitions (see [1,6,16]).

For $n \in N_0 := N \cup \{0\}$, $N = \{1, 2, 3, ...\}$, we define the multiplier transformation D_λ^n of functions $f \in M$ by

$$D_\lambda^n f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+1+\lambda}{\lambda}\right)^n a_k z^k \qquad (\lambda > 0, z \in U^*) .$$

Obviously, we have

$$D_\lambda^m (D_\lambda^n f(z)) = D_\lambda^{m+n} f(z) \qquad (m, n \in N_0; \lambda > 0) .$$

The multiplier transformations D_λ^n and D_1^n were considered ealier by Sarangi and Uralegaddi [15] and Uralegaddi and Somanatha [18, 19], respectively.

Put

$$f_n(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+1+\lambda}{\lambda} \right)^n z^k \qquad (n \in N_0; \lambda > 0) \ .$$

Define the familiar Gaussian hypergeometric function ${}_2F_1(a,b,c;z)$ by

$${}_2F_1(a,b,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \ k!} z^k \qquad (a,b > 0, c \neq 0, -1, ..., z \in U) \ ,$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \begin{cases} 1 & \text{if} \quad k = 0 \\ x(x+1)...(x+k-1) & \text{if} \quad k \in N_0 = \{1, 2, ...\}. \end{cases}$$

Let $f_n^{-1}(z)$ be defined such that

$$f_n(z) * f_n^{-1}(z) = \frac{1}{z} \ {}_2F_1(a,b,c;z) \ .$$

Then we introduce an integral operator $I_\lambda^n(a,b,c) : M \to M$ as follows:

$$I_\lambda^n(a,b,c)f = f_n^{-1}(z) * f(z) \ . \tag{1.2}$$

We note that

$$I_\lambda^0(1,2,1)f(z) = zf^{'}(z) + 2f(z) \ \text{ and } \ I_1^1(1,2,1)f(z) = f(z) \ .$$

It is easily verified from the above definition of the operator $I_\lambda^n(a,b,c)$ that

$$z\left(I_\lambda^{n+1}(a,b,c)f(z)\right)^{'} = \lambda I_\lambda^n(a,b,c)f(z) - (\lambda+1)I_\lambda^{n+1}(a,b,c)f(z) \tag{1.3}$$

and

$$z\left(I_\lambda^n(a,b,c)f(z)\right)^{'} = aI_\lambda^n(a+1,b,c)f(z) - (a+1)I_\lambda^n(a,b,c)f(z) \ . \tag{1.4}$$

The definition (1.2) of the multiplier transformation $I_\lambda^n(1,\mu-1,1) = I_{\lambda,M}^n$[3], is motivated essentially by the Choi-Saigo-Srivastava operator [4], for analytic functions (see also ref. [2]), which includes a simpler integral operator considered earlier by Noor [13] and others (cf. [8], [9], [14]).

By using the integral operator $I_\lambda^n(a,b,c)f$, we introduce the following subclasses of meromorphic functions:

$$MS_\lambda^n(a,b,c;\eta;\varphi) := \{f : f \in M \text{ and } I_\lambda^n(a,b,c)f \in MS(\eta,\varphi)\} \ ,$$

$$MK_\lambda^n(a,b,c;\eta;\varphi) := \{f : f \in M \text{ and } I_\lambda^n(a,b,c)f \in MK(\eta,\varphi)\} \ ,$$

and

$$MC_\lambda^n(a,b,c;\eta,\beta;\varphi,\psi) := \{f : f \in M \text{ and } I_\lambda^n(a,b,c)f \in MC(\eta,\beta,\varphi,\psi)\} \ .$$

We note that

$$f(z) \in MK_\lambda^n(a,b,c;\eta;\varphi) \Rightarrow -zf^{'}(z) \in MS_\lambda^n(a,b,c;\eta;\varphi) \ . \tag{1.5}$$

In particular, we set

$$MS_\lambda^n\left(\eta;\frac{1+Az}{1+Bz}\right) = MS_\lambda^n(\eta;A,B) \quad (-1 < B < A \leq 1) \tag{1.6}$$

and

$$MK_\lambda^n\left(\eta;\frac{1+Az}{1+Bz}\right) =: MK_\lambda^n(\eta;A,B) \quad (-1 < B < A \leq 1) \ . \tag{1.7}$$

The main object of this article is to investigate several properties of the classes $MS_\lambda^n(a,b,c;\eta,\varphi)$, $MK_\lambda^n(a,b,c;\eta,\varphi)$ and $MC_\lambda^n(a,b,c,\eta,\varphi)$ associated with the operator $I_\lambda^n(a,b,c)$ defined by (1.2). Some applications involving integral operator are also considered.

2. Inclusion properties involving the operator $I_\lambda^n(a,b,c)$

The following results will be required in our investigation.

Lemma 1. **[5]** *Let φ be convex univalent in U with $\varphi(0) = 1$ and* $\operatorname{Re}\{\gamma\varphi(z) + w\} > 0$ $(z \in U, \gamma, w \in C)$*. If p is analytic in U with $p(0) = 1$, then the subordinations:*

$$p(z) + \frac{zp^{'}(z)}{\gamma p(z) + w} \prec \varphi(z) \qquad (z \in U)$$

implies that

$$p(z) \prec \varphi(z) \qquad (z \in U) \ .$$

Lemma 2. **[11]** Let φ be convex univalent in U and w be analytic in U with

$$\operatorname{Re}\{w(z)\} \geq 0 \qquad (z \in U) \ .$$

If p is analytic in U and $p(0) = \varphi(0)$, then the subordination:

$$p(z) + w(z)zp^{'}(z) \prec \varphi(z) \qquad (z \in U)$$

implies that

$$p(z) \prec \varphi(z) \qquad (z \in U)\ .$$

First of all, with the help of Lemma 1, we obtain the following inclusion relationships.

Theorem 1. *Let* $\varphi \in N^*$ *with* $\max_{z\in U} \mathrm{Re}\{\varphi(z)\} < \min\left(\frac{a+1-\eta}{1-\eta}, \frac{\lambda+1-\eta}{1-\eta}\right)$ $(a, \lambda > 0; 0 \le \eta < 1)$. *Then*

$$MS_\lambda^n(a+1,b,c;\eta;\varphi) \subset MS_\lambda^n(a,b,c;\eta;\varphi) \subset MS_\lambda^{n+1}(a,b,c;\eta;\varphi)\ .$$

Proof. To prove the first part of Theorem 1, let $f \in MS_\lambda^n(a+1,b,c;\eta;\varphi)$ and set

$$p(z) = \frac{1}{1-\eta}\left(-\frac{z\left(I_\lambda^n(a,b,c)f(z)\right)'}{I_\lambda^n(a,b,c)f(z)} - \eta\right) \tag{2.1}$$

where $p(z) = 1+\gamma_1 z+\gamma_2 z^2+...$ is analytic in U and $p(0) = 1$ for all $z \in U$. Applying (1.4) in (2.1), we obtain

$$-a\frac{I_\lambda^n(a+1,b,c)f(z)}{I_\lambda^n(a,b,c)f(z)} = (1-\eta)p(z) - (a+1-\eta)\ . \tag{2.2}$$

By using the logarithmic differentiation on both side of (2.2), we have

$$\frac{1}{1-\eta}\left(-\frac{z\left(I_\lambda^n(a+1,b,c)f(z)\right)'}{I_\lambda^n(a+1,b,c)f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{a+1-\eta-(1-\eta)p(z)} \tag{2.3}$$

Since

$$\max_{z\in U} \mathrm{Re}\{\varphi(z)\} < \frac{a+1-\eta}{1-\eta} \qquad (z \in U, 0 \le \eta < 1, a > 0)\ ,$$

we see that

$$\mathrm{Re}\{a+1-\eta-(1-\eta)\varphi(z)\} > 0 \qquad (z \in U)\ .$$

Applying Lemma 1 to equation (2.3), it follows that $p \prec \varphi$ in U, that is,

$$f \in MS_\lambda^n(a,b,c,\eta;\varphi)\ .$$

To prove the second inclusion relationship asserted by Theorem 1, let

$$f \in MS_\lambda^n(a,b,c,\eta,\varphi)$$

and put

$$q(z) = \frac{1}{1-\eta}\left(-\frac{z\left(I_\lambda^{n+1}(a,b,c)f(z)\right)'}{I_\lambda^{n+1}(a,b,c)f(z)} - \eta\right)\ ,$$

where the function $q(z)$ is analytic in U with $q(0) = 1$. Then, by using arguments similar to those detailed above with (1.3), it follows that $q \prec \varphi$ in U, which implies that

$$f \in MS_{\lambda}^{n+1}(a, b, c, \eta, \varphi) \ .$$

Thus, we have completed the proof of Theorem 1.

Theorem 2. *Let* $\varphi \in N^*$ *with*

$$\max_{z \in U} \operatorname{Re}\{\varphi(z)\} < \min\left(\frac{a+1-\eta}{1-\eta}, \frac{\lambda+1-\eta}{1-\eta}\right) \quad (a, \lambda > 0, 0 \le \eta < 1) \ .$$

Then

$$MK_{\lambda}^{n}(a+1, b, c; \eta; \varphi) \subset MK_{\lambda}^{n}(a, b, c; \eta; \varphi) \subset MK_{\lambda}^{n+1}(a, b, c; \eta; \varphi) \ .$$

Proof. Applying equation (1.5) and Theorem 1, we observe that

$$f(z) \in MK_{\lambda}^{n}(a+1, b, c; \eta; \varphi) \Leftrightarrow I_{\lambda}^{n}(a+1, b, c) \in MK(\eta, \varphi)$$

$$\begin{aligned}
&\Leftrightarrow \quad -z\left(I_{\lambda}^{n}(a+1, b, c)f\right)^{'} \in MS(\eta, \varphi) \\
&\Leftrightarrow \quad I_{\lambda}^{n}(a+1, b, c)(-zf^{'}(z)) \in MS(\eta, \varphi) \\
&\Leftrightarrow \quad -zf^{'}(z) \in MS_{\lambda}^{n}(a+1, b, c, \eta, \varphi) \\
&\Rightarrow \quad -zf^{'}(z) \in MS_{\lambda}^{n}(a, b, c, \eta, \varphi) \\
&\Leftrightarrow \quad I_{\lambda}^{n}(a, b, c)(-zf^{'}(z)) \in MS(\eta, \varphi) \\
&\Leftrightarrow \quad -z(I_{\lambda}^{n}(a, b, c)f(z))^{'} \in MS(\eta, \varphi) \\
&\Leftrightarrow \quad I_{\lambda}^{n}(a, b, c)f(z) \in MK(\eta, \varphi) \\
&\Leftrightarrow \quad f(z) \in MK_{\lambda}^{n}(a, b, c; \eta; \varphi)
\end{aligned}$$

and

$$f(z) \in MK_{\lambda}^{n}(a, b, c; \eta; \varphi) \Leftrightarrow -zf^{'}(z) \in MS_{\lambda}^{n}(a, b, c; \eta; \varphi)$$

$$\begin{aligned}
&\Rightarrow \quad -zf^{'}(z) \in MS_{\lambda}^{n+1}(a, b, c; \eta; \varphi) \\
&\Leftrightarrow \quad -z(I_{\lambda}^{n+1}(a, b, c)f(z))^{'} \in MS(\eta, \varphi) \\
&\Leftrightarrow \quad I_{\lambda}^{n+1}(a, b, c)f(z) \in MK(\eta, \varphi) \\
&\Leftrightarrow \quad f(z) \in MK_{\lambda}^{n+1}(a, b, c; \eta; \varphi),
\end{aligned}$$

which evidently prove Theorem 2.

By setting

$$\varphi(z) = \frac{1+Az}{1+Bz} \qquad (-1 < B < A \leq 1; z \in U)$$

in Theorem 1 and 2, we deduce the following consequences.

Corollary 1. *Suppose that*

$$\frac{1+A}{1+B} < \min\left(\frac{a+1-\eta}{1-\eta}, \frac{\lambda+1-\eta}{1-\eta}\right) \quad (a, \lambda > 0; 0 \leq \eta < 1; -1 < B < A \leq 1) \ .$$

Then, for the function classes defined by equations (1.6) and (1.7),

$$MS_{\lambda}^{n}(a+1, b, c; \eta; A, B) \subset MS_{\lambda}^{n}(a, b, c; \eta; A, B) \subset MS_{\lambda}^{n+1}(a, b, c; \eta; A, B) \ ,$$

and

$$MK_{\lambda}^{n}(a+1, b, c; \eta; A, B) \subset MK_{\lambda}^{n}(a, b, c; \eta; A, B) \subset MK_{\lambda}^{n+1}(a, b, c; \eta; A, B) \ .$$

Next, by using Lemma 2, we obtain the following inclusion relationships for the class $MC_{\lambda}^{n}(a, b, c; \eta; \beta; \varphi, \psi)$.

Theorem 3. *Let* $\varphi, \psi \in N^*$ *with*

$$\max_{z \in U} \left(\mathrm{Re}\left\{\varphi(z)\right\}\right) < \min\left(\frac{a+1-\eta}{1-\eta}, \frac{\lambda+1-\eta}{1-\eta}\right) \quad (a, \lambda > 0, 0 \leq \eta < 1) \ .$$

Then

$$MC_{\lambda}^{n}(a+1, b, c; \eta, \beta; \varphi, \psi) \subset MC_{\lambda}^{n}(a, b, c; \eta, \beta; \varphi, \psi) \subset MC_{\lambda}^{n+1}(a, b, c; \eta, \beta; \varphi, \psi) \ .$$

Proof. We begin proving that

$$MC_{\lambda}^{n}(a+1, b, c; \eta, \beta; \varphi, \psi) \subset MC_{\lambda}^{n}(a, b, c; \eta, \beta; \varphi, \psi) \ ,$$

which is the first inclusion relationship asserted by Theorem 3. Let

$$f \in MC_{\lambda}^{n}(a+1, b, c; \eta; \beta, \varphi, \psi) \ .$$

Then, in view of the definition of the function class $MC_{\lambda}^{n}(a+1, b, c; \eta, \beta; \varphi, \psi)$, there exists a function $r \in MS(\eta, \varphi)$, such that

$$\frac{1}{1-\beta}\left(-\frac{zI_{\lambda}^{n}(a+1, b, c)f(z))}{r(z)} - \beta\right) \prec \psi(z) \quad (z \in U) \ .$$

Choose the function $g(z)$ such that

$$I_\lambda^n(a+1,b,c)g(z) = r(z) \ .$$

Then

$$g \in MC_\lambda^n(a+1,b,c,\eta;\varphi) \text{ and } \frac{1}{1-\beta}\left(-\frac{z(I_\lambda^n(a+1,b,c)f(z))'}{I_\lambda^n(a+1,b,c)g(z)} - \beta\right) \prec \psi(z) \quad (z \in U) \ . \tag{2.4}$$

Now let

$$p(z) = \frac{1}{1-\beta}\left(-\frac{z(I_\lambda^n(a,b,c)f(z))'}{I_\lambda^n(a,b,c)g(z)} - \beta\right) \ , \tag{2.5}$$

where the function $p(z)$ is analytic in U with $p(0) = 1$. Using equation (1.4), we find that

$$\begin{aligned} &\frac{1}{1-\beta}\left(-\frac{z(I_\lambda^n(a+1,b,c)f(z))'}{I_\lambda^n(a+1,b,c)g(z)} - \beta\right) \\ = \ &\frac{1}{1-\beta}\left(\frac{I_\lambda^n(a+1,b,c)(-zf'(z))}{(I_\lambda^n(a+1,b,c)g(z))} - \beta\right) \\ = \ &\frac{1}{1-\beta}\left(\frac{z(I_\lambda^n(a,b,c)(zf'(z))' + (a+1)I_\lambda^n(a,b,c)zf'(z)}{z(I_\lambda^n(a,b,c)g(z))' + (a+1)I_\lambda^n(a,b,c)g(z)} - \beta\right) \\ = \ &\frac{1}{1-\beta}\left(\frac{\frac{z(I_\lambda^n(a,b,c)(zf'(z))'}{I_\lambda^n(a,b,c)g(z)} + (a+1)\frac{I_\lambda^n(a,b,c)(zf'(z))}{I_\lambda^n(a,b,c)g(z)}}{\frac{z(I_\lambda^n(a,b,c)g(z))'}{I_\lambda^n(a,b,c)g(z)} + a + 1} - \beta\right) \ . \end{aligned} \tag{2.6}$$

Since

$$g \in MS_\lambda^n(a+1,b,c,\eta;\varphi) \subset MS_\lambda^n(a,b,c,\eta;\varphi) \ ,$$

by Theorem 1, we may set

$$q(z) = \frac{1}{1-\eta}\left(-\frac{z(I_\lambda^n(a,b,c)g(z))'}{I_\lambda^n(a,b,c)g(z)} - \eta\right) \ ,$$

where $q(z) \prec \varphi$ in U with the assumption that $\varphi \in N^*$. Then, by virtue of equation (2.5) and (2.6), we observe that

$$I_\lambda^n(a,b,c)(-zf'(z)) = (1-\beta)p(z)I_\lambda^n(a,b,c)g(z) + \beta I_\lambda^n(a,b,c)g(z) \tag{2.7}$$

and

$$\frac{1}{1-\beta}\left(-\frac{z(I_\lambda^n(a+1,b,c)f(z))'}{I_\lambda^n(a+1,b,c)f(z)}-\beta\right)$$

$$= \frac{1}{1-\beta}\left(\frac{z(I_\lambda^n(a,b,c)(-zf'(z))'/I_\lambda^n(a,b,c)g(z)+(a+1)[(1-\beta)p(z)+\beta]}{a+1-\eta-(1-\eta)q(z)}-\beta\right). \quad (2.8)$$

Upon differentiating both sides of equation (2.8), we have

$$\frac{z(I_\lambda^n(a,b,c)(-zf'(z)))'}{I_\lambda^n(a,b,c)g(z)} = (1-\beta)zp'(z) - [(1-\beta)p(z)+\beta]\,[(1-\eta)q(z)+\eta] \;. \quad (2.9)$$

Now, making use of equations (2.4), (2.8), and (2.9), we get

$$\frac{1}{1-\beta}\left(-\frac{z(I_\lambda^n(a+1,b,c)f(z))'}{(I_\lambda^n(a+1,b,c)g(z))}-\beta\right) = p(z) + \frac{zp'(z)}{a+1-\eta-(1-\eta)q(z)} \prec \psi(z) \quad (z\in U). \quad (2.10)$$

since $a > 0$ and $q \prec \varphi$ in U with

$$\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{a+1-\eta}{1-\eta}\ ,$$

we have

$$\operatorname{Re}\{a+1-\eta-(1-\eta)q(z)\} > 0 \qquad (z\in U)\ . \quad (2.11)$$

Hence, by taking

$$w(z) = \frac{1}{a+1-\eta-(1-\eta)q(z)} \quad (2.12)$$

in equation (2.10), and then applying Lemma 2, we can show that $p \prec \psi$ in U, so that

$$f \in MC_\lambda^n(a,b,c,\eta,\beta;\varphi,\psi)\ .$$

For the second inclusion relationship asserted by Theorem 3, using arguments similar to those detailed above with equation (1.3), we obtain

$$MC_\lambda^n(a,b,c,\eta,\beta,\varphi,\psi) \subset MC_\lambda^{n+1}(a,b,c,\eta,\beta;\varphi,\psi)\ .$$

We thus complete the proof of Theorem 3.

3. Inclusion properties involving the integral operator F_c

In this section, we consider the integral operator F_c [see, e.g., ref. [11], pp. 11 and 389] defined by:

$$F_\gamma(f) = F_\gamma(f)(z) := \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma f(t)dt \qquad (f \in M; \gamma > 0) . \tag{3.1}$$

We first state and prove the following inclusion relationship for the integral operator $F_\gamma(\gamma > 0)$.

Theorem 4. *Let $\varphi \in N^*$ with*

$$\max_{z\in U} (\operatorname{Re}\{\varphi(z)\}) < \frac{\gamma+1-\eta}{1-\eta} \qquad (\gamma > 0; 0 \leq \eta < 1) .$$

If

$$f \in MS_\lambda^n(a, b, c, \eta; \varphi) ,$$

then

$$F_\gamma(f) \in MS_\lambda^n(a, b, c, \eta; \varphi) .$$

Proof. Let

$$f \in MS_\lambda^n(a, b, c, \eta; \varphi)$$

and set

$$p(z) = \frac{1}{1-\eta}\left(-\frac{z(I_\lambda^n(a, b, c)F_\gamma(z))'}{I_\lambda^n(a, b, c)F_\gamma(f)(z)} - \eta\right) , \tag{3.2}$$

where the function $p(z)$ is analytic in U with $p(0) = 1$. From the definition (3.1), it is easily verified that

$$z(I_\lambda^n(a, b, c)F_\gamma(f)(z))' = \gamma I_\lambda^n(a, b, c)f(z) - (\gamma + 1)I_\lambda^n(a, b, c)F_\gamma(f)(z) . \tag{3.3}$$

Then, by using equations (3.2) and (3.3), we obtain

$$-\gamma\frac{I_\lambda^n(a, b, c)f(z)}{I_\lambda^n(a, b, c)F_\gamma(f)(z)} = (1-\eta)p(z) - (\gamma + 1 - \eta) . \tag{3.4}$$

Making use of the logarithmic differentiation on both sides of equation (3.4) and multiplying the resulting equation by z, we get

$$\frac{1}{1-\eta}\left(-\frac{z(I_\lambda^n(a, b, c)f(z))'}{I_\lambda^n(a, b, c)f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{\gamma + 1 - \eta - (1-\eta)p(z)} \quad (z \in U) .$$

Hence, by virtue of Lemma 1, we conclude that $p \prec \varphi$ in U for

$$\max\left(\mathrm{Re}\left\{\varphi(z)\right\}\right) < \frac{\gamma+1-\eta}{1-\eta}\ ,$$

which implies that

$$F_\gamma(f) \in MS_\lambda^n(a,b,c,\eta;\varphi)\ .$$

Another inclusion relationship involving the integral operator F_γ $(\gamma > 0)$ is given by Theorem 5 below.

Theorem 5. *Let* $\varphi \in N^*$ *with*

$$\max_{z\in U}\left\{\mathrm{Re}\left(\varphi(z)\right)\right\} < \frac{\gamma+1-\eta}{1-\eta} \qquad (\gamma > 0; 0 \le \eta < 1)\ .$$

If

$$f \in MK_\lambda^n(a,b,c,\eta;\varphi)\ ,$$

then

$$F_\gamma(f) \in MK_\lambda^n(a,b,c,\eta;\varphi)\ .$$

Proof. By applying Theorem 4, it follows that

$$f(z) \in MK_\lambda^n(a,b,c;\eta;\varphi) \Leftrightarrow -zf^{'}(z) \in MS_\lambda^n(a,b,c,\eta;\varphi)$$

$$\begin{aligned} &\Leftrightarrow \quad F_\gamma(-zf^{'}(z)) \in MS_\lambda^n(a,b,c,\eta;\varphi) \\ &\Leftrightarrow \quad -z(F_\gamma(f)(z))^{'} \in MS_\lambda^n(a,b,c,\eta;\varphi) \\ &\Leftrightarrow \quad F_\gamma(f)(z) \in MK_\lambda^n(a,b,c,\eta;\varphi)\ , \end{aligned}$$

which proves Theorem 5.

From Theorem 4 and 5, we can easily deduce Corollary 2 below.

Corollary 2. *Suppose that*

$$\frac{1+A}{1+B} < \frac{\gamma+1-\eta}{1-\eta} \quad (\gamma > 0; -1 < B < A \le 1, 0 \le \eta < 1)\ .$$

Then, for the function class defined by equation (1.3) and (1.4), the following inclusion relationship hold true:

$$f \in MS_\lambda^n(a,b,c,\eta;A,B) \Rightarrow F_\gamma(f) \in MS_\lambda^n(a,b,c,\eta;A,B)$$

and

$$f \in MK_\lambda^n(a,b,c,\eta;A,B) \Rightarrow F_\gamma(f) \in MK_\lambda^n(a,b,c,\eta;A,B)\ .$$

Finally, we prove yet another inclusion relationship involving the integral operator F_γ $(\gamma > 0)$ defined by equation (3.1).

Theorem 6. *Let* $\varphi, \psi \in N^*$ *with*

$$\max_{z\in U}\{\mathrm{Re}\,(\varphi(z))\} < \frac{\gamma+1-\eta}{1-\eta} \qquad (\gamma > 0; 0 \leq \eta < 1)\ .$$

If

$$f \in MC_\lambda^n(a,b,c,\eta,\beta;\varphi,\psi)\ ,$$

then

$$F_\gamma(f) \in MC_\lambda^n(a,b,c,\eta,\beta;\varphi,\psi)\ .$$

Proof. Let

$$f \in MC_\lambda^n(a,b,c,\eta,\beta;\varphi,\psi)\ .$$

Then, in view of the definition of the function class $MC_\lambda^n(a,b,c,\eta,\beta;\varphi,\psi)$, there exists a function $g \in MS_\lambda^n(a,b,c,\eta;\varphi)$ such that

$$\frac{1}{1-\beta}\left(-\frac{z(I_\lambda^n(a,b,c)f(z))'}{I_\lambda^n(a,b,c)g(z)} - \beta\right) \prec \psi(z) \quad (z \in U)\ . \tag{3.5}$$

Thus, we set

$$p(z) = \frac{1}{1-\beta}\left(-\frac{z(I_\lambda^n(a,b,c)F_\gamma(f)(z))'}{I_\lambda^n(a,b,c)F_\gamma(g)(z)} - \beta\right)\ ,$$

where the function $p(z)$ is analytic in U with $p(0) = 1$. Applying equation (3.5), we get

$$\begin{aligned}
&\frac{1}{1-\beta}\left(-\frac{z(I_\lambda^n(a,b,c)f(z))'}{(I_\lambda^n(a,b,c)g(z))} - \beta\right)\\
=\ &\frac{1}{1-\beta}\left(\frac{I_\lambda^n(a,b,c)(-zf'(z))}{(I_\lambda^n(a,b,c)g(z))} - \beta\right)\\
=\ &\frac{1}{1-\beta}\left(\frac{z(I_\lambda^n(a,b,c)F_\gamma(-zf'(z))(z))' + (\gamma+1)I_\lambda^n(a,b,c)F_\gamma(-zf'(z))(z)}{z(I_\lambda^n(a,b,c)F_\gamma(g)(z))' + (\gamma+1)I_\lambda^n(a,b,c)F_\gamma(g)(z)} - \beta\right)
\end{aligned}$$

$$= \frac{1}{1-\beta}\left(\frac{z(I_\lambda^n(a,b,c)F_\gamma(-zf'(z))(z))'/I_\lambda^n(a,b,c)F_\gamma(g)(z) + (\gamma+1)I_\lambda^n(a,b,c)F_\gamma(-zf'(z))(z))'/I_\lambda^n(a,b,c)F_\gamma(g)(z)}{z(I_\lambda^n(a,b,c)F_\gamma(g)(z))'/I_\lambda^n(a,b,c)F_\gamma(g)(z)) + \gamma + 1}\right.$$
$$\left. - \beta\right)\ . \tag{3.6}$$

Since $g \in MS_{\lambda}^{n}(a,b,c,\eta;\varphi)$, we see from Theorem 4 that

$$F_{\gamma}(g) \in MS_{\lambda}^{n}(a,b,c,\eta;\varphi) \ .$$

Let us now put

$$q(z) = \frac{1}{1-\eta}\left(-\frac{z(I_{\lambda}^{n}(a,b,c)F_{\gamma}(g)(z))'}{I_{\lambda}^{n}(a,b,c)F_{\gamma}(g)(z)} - \eta\right) \ , \tag{3.7}$$

where $q \prec \varphi$ in U with the assumption that $\varphi \in N^{*}$. Then, by using the same techniques as in the proof of Theorem 3, we conclude from equations (3.5) and (3.6) that

$$\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda}^{n}(a,b,c)f(z))'}{I_{\lambda}^{n}(a,b,c)g(z)} - \beta\right) = p(z) + \frac{zp'(z)}{\gamma+1-\eta-(1-\eta)q(z)} \prec \psi(z) \quad (z \in U) \ . \tag{3.8}$$

Hence, upon setting

$$w(z) = \frac{1}{\gamma+1-\eta-(1-\eta)q(z)}$$

in equation (3.8), if we apply Lemma 2, we find that $p \prec \psi$ in U, which yields

$$F_{\gamma}(f) \in MC_{\lambda}^{n}(a,b,c,\eta,\beta,\varphi,\psi) \ .$$

The proof of Theorem 6 is evidently completed.

Remark 1. *(1) In their special cases, when* $n = 1,\ \lambda = 1,\ a = 1,\ b = 2$ *and* $c = 1$*, Theorem 4, 5 and 6 would provide extensions of the corresponding results given by Goel and Sohi [6], which reduce to those obtained earlier by Bajpai [1].*

(2)When $a = 1,\ b = \mu,\ c = 1$*, we get the results obtained by [3].*

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