Acta Universitatis Apulensis No. 27/2011
ISSN: 1582-5329 pp. 57-60

SOME REMARKS ON NONCOMMUTATIVE DIRAC EQUATION

Laurian-Ioan Pişcoran

Abstract. In this paper it is continued our previous work (see [1] and [2]) and it is analyzed the noncommutative Dirac equation for the de Broglie harmonic wave function. Also, the noncommutative Dirac equation is presented for the normed wave function which correspond to the n-level of energy in Schrödinger equation.

2000 *Mathematics Subject Classification*: 34L40, 58B99.

1. Introduction

For a particle, the Schrödinger equation, have the following form

$$-\frac{\hbar^2}{2m}\nabla\psi(x,t)+V(x)\psi(x,t)=i\hbar\frac{\partial}{\partial t}\psi(x,t) \tag{1}$$

where $\psi(x,t)$ represent the wave function, $\hbar$ is the Planck constant and m represent the mass of the particle.

If, in the Schrödinger equation, we take $\psi(x,t)=\varphi(x)\exp\left(-\frac{iEt}{\hbar}\right)$ and we introduce this form for the wave function in the equation, then one obtains

$$-\frac{\hbar}{2m}\cdot\frac{d^2}{dx^2}\varphi(x)=[E-V(x)]\varphi(x).$$

The Schrödinger equation for a "particle in the box", have stationary solution of the form: $\psi(x,t)=\varphi(x)\exp\left(-\frac{iEt}{\hbar}\right)$ if the energy E take one of the values E_1, $E_2,\dots,E_n,\dots$ where $E_n=\frac{n^2\cdot\pi^2(\hbar/a)^2}{2m}$, and $n\in\mathbb{Z}_+$.

The normed wave function, which correspond to the n-level of energy take the following form

$$\psi_n(x,t)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{iE_nt}{\hbar}\right). \tag{2}$$

The harmonic de Broglie wave function in one region of constant potential is

$$\psi(x,t) = e^{-iwt}\left(Ae^{ikx} + Be^{-ikx}\right) \tag{3}$$

where A and B are constants and $k = \sqrt{\dfrac{2mE}{\hbar^2}}$.

In our previous papers ([1] and [2]) it is constructed the noncommutative Dirac equation. For a free particle this is of the form (see [2]):

$$\hbar\frac{\partial}{\partial t}\psi(x,t) = -\left(x\partial_x + \frac{1}{2}\right)\psi(x,t).$$

In general, the commutative Dirac equation (see [2]) is

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = Q(x,\partial_x)\psi(x,t) \tag{4}$$

where $Q(x,\partial_x) = \begin{pmatrix} -\dfrac{\alpha\partial_x^2}{2} + \dfrac{\alpha x^2}{2} & -x\partial_x + \dfrac{1}{2} \\ x\partial_x + \dfrac{1}{2} & -\dfrac{\beta\partial_x^2}{2} + \dfrac{\beta x^2}{2} \end{pmatrix}$ with $\alpha, \beta > 0$.

2. Main Results

Theorem 2.1 *For the harmonic de Broglie wave function (3), the noncommutative Dirac equation is*

$$\hbar\, w\, \psi(x,t) = Q(x,\partial_x)\psi(x,t).$$

Proof. From noncommutative Dirac equation (4) one obtain

$$i\hbar\frac{\partial}{\partial t}\left(e^{-iwt}(AE^{ikx} + Be^{-ikx})\right) = \begin{pmatrix} -\dfrac{\alpha\partial_x^2}{2} + \dfrac{\alpha x^2}{2} & -x\partial_x + \dfrac{1}{2} \\ x\partial_x + \dfrac{1}{2} & -\dfrac{\beta\partial_x^2}{2} + \dfrac{\beta x^2}{2} \end{pmatrix}.$$

Hence

$$i\,\hbar(Ae^{ikx} + Be^{-ikx})(-iw)e^{-iwt} = Q(x,\partial_x)\psi(x,t) \Rightarrow \hbar\, w\, \psi(x,t) = Q(x,\partial_x)\psi(x,t).$$

□

Lemma 2.2 *For the normed wave function (2), the noncommutative Dirac equation is*

$$E_n\psi(x,t) = Q(x,\partial_x)\psi(x,t).$$

Proof.
For the normed wave function

$$\psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{iE_n t}{\hbar}\right)$$

if we replace it in the noncommutative Dirac equation, then one obtain

$$i\hbar\frac{\partial}{\partial t}\left(\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{iE_n t}{\hbar}\right)\right) = Q(x,\partial_x)\psi_n(x,t),$$

hence

$$-i^2\hbar\frac{E_n}{\hbar}\,\psi_n(x,t) = Q(x,\partial_x)\psi_n(x,t)$$

hence

$$E_n\psi_n(x,t) = Q(x,\partial_x)\psi_n(x,t).$$

So, the theorem is proved.

□

Corollary 2.3 *If we consider the normed wave function (2) using the noncommutative Dirac equation (4), one obtain the total energy* $E_n = \frac{1}{\hbar}\left(px + \frac{\hbar}{2i}\right)$.
Proof.

$$\hbar\frac{\partial}{\partial t}\,\psi_n(x,t) = -\left(x\partial_x + \frac{1}{2}\right)\psi_n(x,t) \Rightarrow -\frac{i\hbar}{\hbar}\,E_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{iE_n t}{\hbar}\right)$$
$$= -\left(x\partial_x + \frac{1}{2}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{iE_n t}{\hbar}\right) \Rightarrow iE_n = x\partial_x + \frac{1}{2}$$
$$\Rightarrow E_n = \frac{1}{i}\left(x\partial_x + \frac{1}{2}\right) \Rightarrow E_n = \frac{1}{\hbar}\left(px + \frac{\hbar}{2i}\right).$$

We replace the total energy E_n in the normed wave function and obtain

$$\psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{i\left(px + \frac{\hbar}{2i}\right)t}{\hbar^2}\right) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{i\,p\,x\,t}{\hbar^2} + \frac{t}{2i\hbar}\right).$$

□

Corollary 2.4 *Consider the harmonic de Broglie function (3). Using the noncommutative Dirac equation (4), one obtain the total energy*

$$E = \frac{1}{2m\,w}\left(px + \frac{\hbar}{2i}\right).$$

Proof. Using the equality $\frac{1}{i}\left(x\partial_x + \frac{1}{2}\right) = \frac{1}{\hbar}\left(px + \frac{\hbar}{2i}\right)$ it follows

$$\frac{2m}{\hbar^2}\,E\,\hbar\,w = \frac{1}{\hbar}\left(px + \frac{\hbar}{2i}\right) \Rightarrow E = \frac{1}{2m\,w}\left(px + \frac{\hbar}{2i}\right).$$

□

Next, we will consider a plane wave $\psi(\overrightarrow{x}, t) = Cexp\left(\frac{i}{\hbar}p_i x - \frac{i}{\hbar}Et\right)$ which represent a particle A which is incident with a B particle. Here p_i represent the impulse of the wave, E is the energy and C is a constant. The form of the wave function which describe the diffracted wave is: $\psi_s(x,t) = Cf(\theta)\frac{1}{x}exp\left(\frac{i}{\hbar}(px - Et)\right)$ The function $f(\theta)$ is a application betwen the direction of the impulse p_i and the direction of the vector x.

Theorem 2.5 *For the wave function $\psi_s(x,t)$, the noncommutative Dirac equation is*

$$-iE\psi_s(x,t) = Q(x, \partial_x)\psi_s(x,t).$$

Proof. From $\hbar\frac{\partial}{\partial t}\,\psi_s(x,t) = -Q(x, \partial(x))\psi_s(x,t)$ one obtain:
$Cf(\theta)\frac{1}{x}(-E)exp\left(\frac{i}{\hbar}(px - Et)\right) = Q(x, \partial(x))\psi_s(x,t) \Rightarrow -iE\psi_s(x,t) = Q(x, \partial_x)\psi_s(x,t).$

□

References

[1] Pişcoran, L., *The Dirac equation and the noncommutative harmonic oscilator*, Studia Univ. "Babeş-Bolyai", Mathematica, **LIII** (2008), no. 2, 79–83.

[2] Pişcoran, L., *The noncommutative Dirac equation for some wave functions*, Acta Univ. Apullensis, Alba Iulia, **13** (2007), 71–74.

[3] Thaller, B., *The Dirac equation*, Springer-Verlag, Berlin, Heidelberg, New-York, 1992.

[4] Varilly, J. and Bondia, J. G., *Elements of noncommutative geometry*, Birkhauser, Berlin, 2001

Laurian-Ioan Pişcoran
Department of Mathematics and Computer Science
University of Baia Mare, Romania
Address Str. Victoriei Nr.76, 430122
email:*plaurian@yahoo.com*