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CERTAIN CLASSES OF MULTIVALENT FUNCTIONS RELATED WITH A LINEAR OPERATOR

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Abstract. In this paper, we introduce and study some new classes of analytic functions using a convolution operator $L_p^*(a,c) : A \to A$. Some inclusion relationships and a radius problem are investigated. We also show that the class $R_{k,p}(a,c,\alpha)$ is closed under convolution operator with a convex function for $k = 2$.

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1.Introduction

Let A_p denotes the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \tag{1}$$

which are analytic in the open unit disc E=$\{z : |z| < 1\}$ and $p \in N = \{1, 2, 3, 4, ...\}$. Further for $0 \le \alpha < p$, we denote $S_p^*(\alpha)$, $C_p(\alpha)$ and $K_p(\alpha, \gamma)$ be the sbclasses of A_p consisting of functions which are respectively, p-valently starlike, convex and close-to-convex of order α and type γ in E. For $\alpha = 0$ these classes S_p^* and K_p was introduced by Goodman [2].

The convolution (or Hadmard product) is deonoted and defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}\ b_{p+k}\ z^{p+k}, \tag{1.2}$$

where

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad \text{and} \quad g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}.$$

The generalized Bernadi operator is deonoted and defined as,

$$J_{c,p}(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\ dt,\ c > -p. \tag{1.3}$$

Inspiring from carlson Shaffer, Saitoh [9] introduced a linear operator, $L_p(a,c)$, $(a \in R, c \in C - \{0,-1,-2,...\})$ as:

$$L_p(a,c) f(z) = \phi_p(a,c;z) * f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+p}\ z^{p+k}, \tag{1.4}$$

where

$$\phi_p(a,c;z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k}$$

and $(a)_k$ is Pochhammer symbol.

Al-Kharasani and Al-Hajiry [**1**] defined the linear operator $L_p^*(a,c)$ as

$$L_p^*(a,c) f(z) = \phi_p^*(a,c;z) * f(z), \tag{1.5}$$

where

$$\phi_p(a,c;z) * \phi_p^*(a,c;z) = \frac{z^p}{(1-z)^{p+1}}. \tag{1.6}$$

From (1.5) and (1.6) the following identity can be easily verified

$$L_p^*(a,c+1) f(z) = z \left(L_p^*(a,c) f(z)\right)' + (c-p) L_p^*(a,c) f(z). \tag{1.7}$$

Let $P_k(\alpha)$ be the class of functions $p(z)$ analytic in the unit disc E, satisfying the properties $p(0) = 1$ and

$$\int_0^{2\Pi} \left| \frac{\text{Re} p(z) - \alpha}{1-\alpha} \right| d\theta \leq k\pi,$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \alpha < 1$. For $\alpha = 0$, we obtain the class P_k defined by Pinchuk [6].

We also represent $p \in P_k(\alpha)$ as

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \tag{1.8}$$

where $p_i \in P(\alpha)$, for $i = 1,2$ and $z \in E$.

We have the following known classes. For $0 \leq \alpha,\ \beta < 1,\ k \geq 2$,

$R_k(\alpha) = \left\{ f : f \in A_p \text{ and } \frac{zf'}{f} \in P_k(\alpha) \right\}$,

$V_k(\alpha) = \left\{ f : f \in A_p \text{ and } \frac{(zf')'}{f'} \in P_k(\alpha) \right\}$ and

$T_k(\beta, \alpha) = \left\{ f : f \in A_p, g \in R_2(\alpha) \text{ and } \frac{zf'}{g} \in P_k(\beta) \right\}$.

Remark 1.1.

$$f \in V_k(\alpha) \iff \frac{zf'}{p} \in R_k(\alpha).$$

Using the operator $L_p^*(a, c)$, we introduce the following new classes of analytic functions. For $0 \leq \alpha,\ \beta < 1,\ k \geq 2$.

Definition 1.1.
$R_{k,p}(a, c, \alpha) = \left\{ f : f \in A_p \text{ and } L_p^*(a, c) f \in R_k(\alpha) \right\}$.

Definition 1.2.
$V_{k,p}(a, c, \alpha) = \left\{ f : f \in A_p \text{ and } L_p^*(a, c) f \in V_k(\alpha) \right\}$.

Definition 1.3.
$T_{k,p}(a, c, \beta, \alpha) = \left\{ f : f \in A_p \text{ and } L_p^*(a, c) f \in T_k(\beta, \alpha) \right\}$.

Note.

For special values of parameters these classes were investigated by several authors, see [1-3]and [5].

2.Preliminary Results

Lemma 2.1 [4]. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let ϕ be a complex-valued function satisfying the conditions.

i) $\phi(u, v)$ is continuous in $D \subset C^2$,

ii) $(1, 0) \in D$ and $\text{Re}\phi(1, 0) > 0$,

iii) $\text{Re}\phi(iu_2, v_1) \leq 0$, where $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}\left(1 + u_2^2\right)$.

If

$$h(z) = 1 + \sum_{m=2}^{\infty} c_m z^m,$$

is a function analytic in E such that $\left(h(z), zh'(z)\right) \in D$ and $\text{Re}\phi\left(h(z), zh'(z)\right) > 0$, for $z \in E$, then $\text{Re}h(z) > 0$ in E.

Lemma 2.2

Let $p(z)$ be analytic in E with $p(0)=1$ and $\mathrm{Re}p(z)>0$, $z\in E$. Then for $s>0$ and $\eta\neq 1$ (complex),

$\mathrm{Re}\left\{p(z)+\frac{szp'(z)}{p(z)+\eta}\right\}>0$, for $|z|<r_\circ$,

where $r_\circ$ is given by

$$r_\circ=\frac{|\eta+1|}{\sqrt{A+(A^2-|\eta^2-1|)^{\frac{1}{2}}}},\ A=2(s+1)^2+|\eta|^2-1$$

and this radius is best possible. For this result we refer to [8].

Lemma 2.3[7]

Let ψ be convex and g be starlike in E. Then, for F analytic in E with $F(0)=1$, $\frac{\psi*Fg}{\psi*g}$ is contained in the convex hull of $F(E)$.

Lemma 2.4[10]

If $p(z)$ is analytic in E with $p(0)=1$, and if λ is a complex number satisfying $\mathrm{Re}\lambda\geq 0, (\lambda\neq 0)$, then $\mathrm{Re}\left[p(z)=\lambda zp'(z)\right]>\beta$, $(0\leq\beta<1)$ implies $\mathrm{Re}p(z)>\beta+(1-\beta)(2\gamma-1)$, where γ is given by

$$\gamma=\int_0^1\left(1+t^{\mathrm{Re}\lambda}\right)^{-t}dt,$$

which is an increasing function of $\mathrm{Re}\lambda$ and $\frac{1}{2}\leq\gamma<1$. The estimate is sharp in the sense that bound cannot be improved.

3.Main results

Theorem 3.1. For $0\leq\alpha<p$, $c\geq p$, $k\geq 2$,

$$R_{k,p}(a,c+1,\alpha)\subseteq R_{k,p}(a,c,\beta),$$

where

$$\beta=\frac{2[p-2\alpha(p-c)]}{\sqrt{(2c-2p-2\alpha+1)^2+8(p-2\alpha(p-c))+(2c-2p-2\alpha+1)}}. \tag{3.1}$$

Proof. Let $f\in R_{k,p}(a,c+1,\alpha)$ and let

$$\frac{z\left(L_p^*(a,c)f(z)\right)'}{L_p^*(a,c)f(z)}=H(z)=(p-\beta)h(z)+\beta. \tag{3.2}$$

From $(1.7), (3.2)$ and after some simplification, we have

$$\frac{z\left(L_p^*(a,c+1)\,f(z)\right)'}{L_p^*(a,c+1)\,f(z)}-\alpha=(\beta-\alpha)+(p-\beta)\,h(z)+\frac{(p-\beta)zh'(z)}{(p-\beta)\,h(z)+(\beta+c-p)} \tag{3.3}$$

and

$$H(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \tag{3.4}$$

where $h(z)=1+c_1z+c_2z^2+...$ is analytic and $h(0)=1$ in E. We want to show that $H(z)\in P_k(\beta)$ or $h_i(z)\in P(\beta)$, $i=1,2$.

From $(3.3), (3.4)$ we have

$$\begin{aligned}\frac{z\left(L_p^*(a,c+1)f(z)\right)'}{L_p^*(a,c+1)f(z)}-\alpha &= \left(\tfrac{k}{4}+\tfrac{1}{2}\right)\{(\beta-\alpha)+(p-\beta)\,h_1(z)\}\\ &-\left(\tfrac{k}{4}-\tfrac{1}{2}\right)\{(\beta-\alpha)+(p-\beta)\,h_2(z)\}\\ &+\left(\tfrac{k}{4}+\tfrac{1}{2}\right)\left\{\tfrac{(p-\beta)zh_1'(z)}{(p-\beta)h_1(z)+(\beta+c-p)}\right\}\\ &-\left(\tfrac{k}{4}-\tfrac{1}{2}\right)\left\{\tfrac{(p-\beta)zh_2'(z)}{(p-\beta)h_2(z)+(\beta+c-p)}\right\}\\ =\left(\tfrac{k}{4}+\tfrac{1}{2}\right)&\left\{(\beta-\alpha)+(p-\beta)\,h_1(z)+\tfrac{(p-\beta)zh_1'(z)}{(p-\beta)h_1(z)+(\beta+c-p)}\right\}\\ -\left(\tfrac{k}{4}-\tfrac{1}{2}\right)&\left\{(\beta-\alpha)+(p-\beta)\,h_2(z)+\tfrac{(p-\beta)zh_2'(z)}{(p-\beta)h_2(z)+(\beta+c-p)}\right\}\end{aligned}$$

and this implies that

$$\mathrm{Re}\left\{(\beta-\alpha)+(p-\beta)\,h_i(z)+\frac{(p-\beta)\,zh_i'(z)}{(p-\beta)\,h_1(z)+(\beta+c-p)}\right\}>0, z\in E, i=1,2.$$

We formulate a functional $\phi(u,v)$ by taking $u=h_i\ (z)$ and $v=zh_i'(z)$. Thus

$$\phi(u,v)=(\beta-\alpha)+(p-\beta)\,h_i(z)+\frac{(p-\beta)\,zh_i'(z)}{(p-\beta)\,h_1(z)+(\beta+c-p)}.$$

It can be easily seen that $\phi(u,v)$ satisfies the conditions (i) and (ii) of Lemma (2.1) in the domain of $D\subseteq C\times(C-\frac{\beta+c-p}{\beta-p})$.

To verify the condition (iii) we proceed as follows

$$\mathrm{Re}[\phi(iu_2,v_1)]=(\beta-\alpha)+\frac{(p-\beta)\,(\beta+c-p)}{(p-\beta)^2\,u_2^2+(\beta+c-p)^2}.$$

When we put $v_1 \leq -\frac{1}{2}\left(1+u_2^2\right)$, then $\text{Re}[\phi\left(iu_2, v_1\right)] \leq \frac{A+Bu_2^2}{2C}$, where

$$A = 2\left(\beta-\alpha\right)\left(\beta = c-p\right)^2 - \left(\beta+c-p\right)\left(p-\beta\right),$$

$$B = 2\left(\beta-\alpha\right)\left(p-\beta\right)^2 - \left(\beta+c-p\right)\left(p-\beta\right),$$

$$C = \left(\beta = c-p\right)^2 + \left(p-\beta\right)^2 u_2^2 > 0.$$

Note that $\text{Re}[\phi\left(iu_2, v_1\right)] \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$ we obtain β, given by (3.1) and from $B \leq 0$ gives $0 \leq \beta < p$. Hence $h_i\left(z\right) \in P\left(\beta\right)$ and consequently $f\left(z\right) \in R_{k,p}\left(a, c, \beta\right)$.

Theorem 3.2. For $0 \leq \alpha < p$, $c \geq p$, $k \geq 2$, where β is given by (3.1),

$$V_{k,p}\left(a, c+1, \alpha\right) \subseteq V_{k,p}(a, c, \alpha).$$

*Proof.*Let $f \in V_{k,p}(a, c+1, \alpha)$. Then $L_p^*\left(a,c\right) f\left(z\right) \in V_k\left(\alpha\right)$ and by remark (1.1), we have $z\left(L_p^*\left(a,c\right) f\left(z\right)\right)' \in R_k\left(\alpha\right)$. This implies $L_p^*\left(a,c\right)\left(zf'\left(z\right)\right) \in R_k\left(\alpha\right) \implies zf'\left(z\right) \in R_{k,p}\left(a, c+1, \alpha\right) \subseteq R_{k,p}\left(a, c, \alpha\right)$. Consequently $f \in V_{k,p}\left(a, c, \alpha\right)$, where β is given by (3.1).

Theorem 3.3. For $0 \leq \alpha < p$, $c \geq p$, $k \geq 2$,

$$T_{k,p}\left(a, c+1, \beta, \alpha\right) \subseteq T_{k,p}\left(a, c, \beta, \alpha\right).$$

Proof. Let $f \in T_{k,p}\left(a, c+1, \beta, \alpha\right)$. Then there exists $g_1\left(z\right) \in R_2\left(\alpha\right)$ such that

$$\frac{z\left(L_p^*(a, c+1)\, f\left(z\right)\right)'}{g_1(z)} \in P_k\left(\beta\right). \tag{3.5}$$

Let $g_1\left(z\right) = L_p^*\left(a, c+1\right) g\left(z\right)$. Then $g\left(z\right) \in R_{2,p}\left(a, c+1, \alpha\right)$.

We set

$$\frac{z\left(L_p^*(a, c)\, f\left(z\right)\right)'}{L_p^*\left(a,c\right) g(z)} = H\left(z\right) = \left(p-\beta\right) h\left(z\right) + \alpha, \tag{3.6}$$

where $h\left(z\right) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ is analytic and $h\left(0\right) = 1$ in E. By using (1.7) and after some simplification, we get

$$\frac{z\left(L_p^*(a, c+1)\, f\left(z\right)\right)'}{L_p^*\left(a, c+1\right) g(z)} = \frac{z\left(L_p^*(a, c)\left(zf'\left(z\right)\right)\right)' + \left(c-p\right) L_p^*\left(a,c\right)\left(zf'\left(z\right)\right)}{z(L_p^*\left(a,c\right) g(z))' + \left(c-p\right) L_p^*\left(a,c\right) g(z)}. \tag{3.7}$$

Also, $g \in R_{2,p}\left(a, c+1, \alpha\right)$ and by using Theorem (3.1), with $k = 2$ and $\beta = \alpha$, we have $g \in R_{2,p}\left(a, c, \alpha\right)$.Therefore we can write

$$\frac{z\left(L_p^*(a, c)\, g\left(z\right)\right)'}{L_p^*\left(a,c\right) g(z)} = H_\circ\left(z\right) = \left(p-\alpha\right) q\left(z\right) + \alpha, \text{ where } q\left(z\right) \in P.$$

By logarithmic differentiation of (3.6) and after some simplification, we have

$$\frac{z\left(L_p^*(a,c)\left(zf'\left(z\right)\right)\right)'}{L_p^*\left(a,c+1\right)g(z)} = H\left(z\right)H_\circ\left(z\right) + zH'\left(z\right). \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\frac{z\left(L_p^*(a,c+1)\,f\left(z\right)\right)'}{L_p^*\left(a,c+1\right)g(z)} = H\left(z\right) + \frac{zH'\left(z\right)}{H_\circ\left(z\right)+\left(c-p\right)} \in P_k\left(\beta\right). \tag{3.9}$$

Let

$$H\left(z\right) = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{\left(p-\beta\right)h_1\left(z\right)+\alpha\right\} - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{\left(p-\beta\right)h_2\left(z\right)+\alpha\right\}$$

and
$c\left(z\right) = H_\circ\left(z\right)+\left(c-p\right) = \left(p-\alpha\right)q\left(z\right)+\alpha+\left(c-p\right).$

We want to show that $H \in P_k\left(\beta\right)$ or $h_i \in P$ for $i=1,2$. Then $\mathrm{Re}\,c\left(z\right) > 0$ if $\left(c-p\right) > -\alpha$.

From (3.6) and (3.7), we will get

$$\begin{aligned}\frac{z\left(L_p^*(a,c+1)\,f\left(z\right)\right)'}{L_p^*\left(a,c+1\right)g(z)} - \alpha &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{\left(p-\beta\right)h_1\left(z\right)+\frac{\left(p-\beta\right)zh_1'\left(z\right)}{\left(p-\alpha\right)q\left(z\right)+\alpha+\left(c-p\right)}\right\}\\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\left(p-\beta\right)h_2\left(z\right)+\frac{\left(p-\beta\right)zh_2'\left(z\right)}{\left(p-\alpha\right)q\left(z\right)+\alpha+\left(c-p\right)}\right\}\end{aligned}$$

and this implies that

$$\mathrm{Re}\left\{\left(p-\beta\right)h_i\left(z\right)+\frac{\left(p-\beta\right)zh_i'\left(z\right)}{\left(p-\alpha\right)q\left(z\right)+\alpha+\left(c-p\right)}\right\} > 0, z\in E, i=1,2.$$

Now by taking $u = h_i\left(z\right)$ and $v = zh_i'\left(z\right)$, we formulate a functional $\phi\left(u,v\right)$. Thus

$$\phi\left(u,v\right) = \left(p-\beta\right)u + \frac{\left(p-\beta\right)v}{\left(p-\alpha\right)q\left(z\right)+\alpha+\left(c-p\right)}.$$

Then clearly $\phi\left(u,v\right)$ satisfies the conditions (i) and (ii) of Lemma (2.1).

To verify the condition (iii), we start, with $q\left(z\right) = q_1 + iq_2$, as follows:

$$\begin{aligned}\mathrm{Re}[\phi\left(iu_2,v_1\right)] &= \mathrm{Re}\left\{\frac{\left(p-\beta\right)v_1}{\left(p-\alpha\right)\left(q_1+iq_2\right)+\alpha+\left(c-p\right)}\right\}\\ &= \frac{\left(p-\beta\right)\left\{\left(p-\alpha\right)q_1+\alpha+\left(c-p\right)\right\}v_1}{\left\{\left(p-\alpha\right)q_1+\alpha+\left(c-p\right)\right\}^2+\left(p-\alpha\right)^2q_2^2}.\end{aligned}$$

After putting $v_1 \leq -\frac{1}{2}\left(1+u_2^2\right)$, we have

$$\operatorname{Re}[\phi(iu_2, v_1)] \leq \frac{(p-\beta)\{(p-\alpha)q_1+\alpha+(c-p)\}v_1}{\{(p-\alpha)q_1+\alpha+(c-p)\}^2+(p-\alpha)^2 q_2^2} \leq 0.$$

By applying Lemma (2.1), we have $\operatorname{Re}h_i(z) > 0$, for $i = 1,2$ and consequently $h(z) \in P$. Thus $f \in T_{k,p}(a,c,\beta,\alpha)$.

Theorem 3.4. Let $f \in R_{k,p}(a,c,\beta)$, then $J_{c,p}f \in R_{k,p}(a,c,\beta)$.

Proof. Let

$$\frac{z\left(L_p^*(a,c)\,J_{c,p}f(z)\right)'}{L_p^*(a,c)\,J_{c,p}f(z)} = H(z) = (p-\beta)h(z)+\beta. \tag{3.10}$$

where $h(z) = 1+c_1z+c_2z^2+c_3z^3+...$ is analytic and $h(0)=1$ in E. Using (1.7),(3.10) and after some simplification, we have

$$\frac{z\left(L_p^*(a,c)\,J_{c,p}f(z)\right)'}{L_p^*(a,c)\,J_{c,p}f(z)} = (p-\beta)h(z)+\frac{(p-\beta)zh'(z)}{(p-\beta)h(z)+(\beta+c)} \in P_k(\beta) \tag{3.11}$$

and

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z). \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\begin{aligned}\frac{z\left(L_p^*(a,c)\,J_{c,p}f(z)\right)'}{L_p^*(a,c)\,J_{c,p}f(z)} &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\beta)h_1(z)+\frac{(p-\beta)zh_1'(z)}{(p-\beta)h_1(z)+(\beta+c))}\right\}\\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\beta)h_2(z)+\frac{(p-\beta)zh_2'(z)}{(p-\beta)h_1(z)+(\beta+c))}\right\}\end{aligned}$$

and this implies that

$$\operatorname{Re}\left\{(p-\beta)h_i(z)+\frac{(p-\beta)zh_i'(z)}{(p-\beta)h_i(z)+(\beta+c)}\right\} > 0, z \in E, i = 1,2.$$

Now we define a function $\phi(u,v)$, by putting $u = h_i(z)$ and $v = zh_i'(z)$.

Thus

$$\phi(u,v) = (p-\beta)u+\frac{(p-\beta)v}{(p-\beta)u+(\beta+c)}.$$

Then clearly $\phi(u,v)$ satisfies all the propereties of Lemma (2.1). Hence $H(z) \in P_k(\beta)$ and consequently $J_{c,p}f \in R_{k,p}(a,c,\beta)$.

Theorem 3.5. Let ϕ be a convex function and $f \in R_{2,p}(a,c,\alpha)$. Then $G \in R_{2,p}(a,c,\alpha)$, where $G = \phi * f$.

Proof. Let $G = \phi * f$ and let

$$\phi(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \quad , \qquad f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}.$$

Then

$$L_p^*(a,c)\, G = \phi * \left(L_p^*(a,c)\, f\right). \tag{3.13}$$

Also, $f \in R_{2,p}(a,c,\alpha)$. Therefore, $L_p^*(a,c)\, f \in R_{2,}(\alpha) = S_p^*(\alpha)$.

By logarithmic differentiation of (3.13) and after some simplification, we have

$$\frac{z(L_p^*(a,c)\, G)'}{L_p^*(a,c)\, G} = \frac{\phi * F L_p^*(a,c)\, f}{\phi * L_p^*(a,c)\, f},$$

where

$$F = \frac{z\left(L_p^*(a,c)\, f\right)'}{L_p^*(a,c)\, f}.$$

As $F(z)$ is analytic in E and $F(0) = 1$. From Lemma (2.3), we can see that $\frac{z(L_p^*(a,c)G)'}{L_p^*(a,c)G}$ is contained in the convex hull of $F(E)$. Since $\frac{z(L_p^*(a,c)G)'}{L_p^*(a,c)G}$ is analytic in E and

$$F(E) \subseteq \Omega = \left\{ W : \frac{z(L_p^*(a,c)\, W(z))'}{L_p^*(a,c)\, W(z)} \in P_2(\alpha) \right\},$$

then $\frac{z(L_p^*(a,c)G)'}{L_p^*(a,c)G}$ lies in Ω. This implies that $G = \phi * f \in R_{2,p}(a,c,\alpha)$.

Theorem 3.6. Let for $z \in E$, $f(z) \in R_{k,p}(a,c,\alpha)$ .Then $f(z) \in R_{k,p}(a,c+1,\alpha)$, for

$$|z| < r_\circ = \frac{|\eta + 1|}{\sqrt{A + (A^2 - |\eta^2 - 1|)^{\frac{1}{2}}}}, \tag{3.14}$$

where $A = 2(s+1)^2 + |\eta|^2 - 1$, with $\eta = \frac{\alpha + p - c}{p - \alpha}$, and $s = \frac{1}{p-\alpha}$. The value of $r_\circ$ is exact.

Proof. Let

$$\frac{z\left(L_p^*(a,c)\, f(z)\right)'}{L_p^*(a,c)\, f(z)} = H(z) = (p-\alpha)\, h(z) + \alpha. \tag{3.15}$$

Where $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ is analytic and $h(0) = 1$ in E. From (1.7), (3.15) and after some simplification, we have

$$\left(\frac{1}{p-\alpha}\right) \left\{ \frac{z\left(L_p^*(a,c+1)\, f(z)\right)'}{L_p^*(a,c+1)\, f(z)} - \alpha \right\} = h(z) + \frac{\left(\frac{1}{p-\alpha}\right) z h'(z)}{h(z) + \frac{\alpha+p-c}{p-\alpha}}. \tag{3.16}$$

Also

$$H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z)$$

and (3.16) implies

$$\begin{aligned} \left(\frac{1}{p-\alpha}\right)\left\{\frac{z\left(L_p^*(a,c+1)\,f(z)\right)'}{L_p^*(a,c+1)\,f(z)} - \alpha\right\} &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{\left(\frac{1}{p-\alpha}\right) z h_1'(z)}{h_2(z) + \frac{\alpha+p-c}{p-\alpha}}\right\} \\ &\quad - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z) + \frac{\left(\frac{1}{p-\alpha}\right) z h_2'(z)}{h_2(z) + \frac{\alpha+p-c}{p-\alpha}}\right\}. \end{aligned}$$

where $\mathrm{Re}h_i(z) > 0$, for $i = 1, 2$. By using Lemma (2.2), with $\eta = \frac{\alpha+p-c}{p-\alpha} \neq -1$, and $s = \frac{1}{p-\alpha}$, we have

$$\mathrm{Re}\left\{h_i(z) + \frac{szh_i'(z)}{h_i(z)\,\eta}\right\} > 0, \text{ for } |z| < r_\circ,$$

and $r_\circ$ is given by (3.14). Thus $\frac{z\left(L_p^*(a,c)f(z)\right)'}{L_p^*(a,c)f(z)} \in P_k(\alpha)$ and consequently $f(z) \in R_{k,p}(a, c+1, \alpha)$ for $|z| < r_\circ$ and this radius is best possible.

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