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ON CURVATURE INHERITING SYMMETRY IN FINSLER SPACE

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Abstract. K. L. Duggal[3] has stuided curvature Inheritance symmetry Riemannian space with application of fluid space time. Ricci curvature inheriting symmetry of semi-Riemannian manifolds were introduced by K. L. Duggal and R. Sharma[7]. In this paper we have study on curvature inheritance symmetry and Ricci-Inheriting symmetry in Finsler space and investigated some results.

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1.Introduction and Preliminaries

We consider an n-dimensional Finsler space (Fn) in which the curvature tensor field due to Berwald's are given by

$$H^i_{jk} = \frac{\partial^2 G^i}{\partial x^k \partial \dot{x}^j} - \frac{\partial^2 G^i}{\partial x^j \partial \dot{x}^k} + G^i_{kr}\frac{\partial G^r}{\partial \dot{x}^j} - G^i_{rj}\frac{\partial G^r}{\partial \dot{x}^k}, \tag{1}$$

$$H^i_{jkh} = \frac{\partial G^i_{jk}}{\partial x^h} - \frac{\partial G^i_{jh}}{\partial x^k} + G^r_{jh}G^i_{rk} - G^r_{jk}G^i_{rh} + G^i_{rjh}\frac{\partial G^r}{\partial \dot{x}^k} - G^i_{rjk}\frac{\partial G^r}{\partial \dot{x}^h}, \tag{2}$$

where

$$\begin{cases} (a) & H^i_{jkh} = \frac{\partial H^i_{kh}}{\partial \dot{x}^j}, \\ (b) & G^i_{jkh} = \frac{\partial G^i_{kh}}{\partial \dot{x}^j}. \end{cases} \tag{3}$$

The commutation formulae involving the curvature tensor are given by

$$T_{(h)(k)} - T_{(k)(h)} = \frac{\partial T}{\partial \dot{x}^i} H^i_{hk}, \tag{4}$$

$$T_{ij(h)(k)} - T_{ij(k)(h)} = -\frac{\partial T_{ij}}{\partial \dot{x}^r} H^r_{hk} - T_{rj} H^r_{ihk} - T_{ir} H^r_{jhk}, \tag{5}$$

$$\begin{aligned} T^i_{jkh(l)(m)} - T^i_{jkh(l)(m)} &= -\frac{\partial T^i_{jkh}}{\partial \dot{x}^r} H^r_{lm} + T^r_{jkh} H^i_{rlm} - T^i_{rkh} H^r_{jlm} \\ &\quad - T^i_{jrh} H^r_{klm} - T^i_{jkr} H^r_{hlm}. \end{aligned} \tag{6}$$

The curvature tensor satisfies the following relations.

$$\begin{cases} (a) & H^i_{jkh} = -H^i_{jhk}, \\ (b) & H^i_{jkh} = \dot{\partial}_j H^i_{kh}, \\ (c) & H_{kh} = H^i_{ikh}, \\ (d) & H_k = H^i_{ik}, \\ (e) & H^j_j = (n-1)H, \\ (f) & \dot{\partial}_l H^i_{jkh} \dot{x}^j = \dot{\partial}_l H^i_{jkh} \dot{x}^l = 0, \\ (g) & H^i_{jkh} \dot{x}^h = H^i_{jk}, \\ (h) & H^i_{jk} \dot{x}^h = H^i_j, \\ (i) & H^i_j \dot{x}^j = 0. \end{cases} \tag{7}$$

The relative curvature tensor in a Finsler space is defined as[1].

$$\begin{aligned} \tilde{K}^i_{jkh} &= \left(\frac{\partial \Gamma^{*i}_{jk}}{\partial x^h} + \frac{\partial \Gamma^{*i}_{jk}}{\partial \dot{x}^i} \frac{\partial \xi^i}{\partial x^h} \right) - \left(\frac{\partial \Gamma^{*i}_{jh}}{\partial x^k} + \frac{\partial \Gamma^{*i}_{jh}}{\partial \dot{x}^l} \frac{\partial \xi^l}{\partial x^k} \right) \\ &\quad + \Gamma^{*i}_{mh} \Gamma^{*m}_{jk} + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh}. \end{aligned} \tag{8}$$

The commutation formula involving the relative curvature tensor is given by

$$T_{ij;kh} - T_{ij;hk} = -T_{ir} \tilde{K}^i_{jkh} - T_{rj} \tilde{K}^r_{ikh}, \tag{9}$$

where T_{ij} is an arbitrary tensor field and ; is a covariant δ derivative.
The relative curvature tensor satisfies the following relations.

$$\begin{cases} (a) & \tilde{K}^i_{jkh} = -\tilde{K}^i_{jhk}, \\ (b) & \tilde{K}_{ij} = \tilde{K}^l_{ijl}. \end{cases} \tag{10}$$

Let an infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon v^i(x^j), \tag{11}$$

be generated by a vector field $v^i(x^j)$ independent of directional arguments (dependent of positional coordinates x^i only). This transformation is called infinitesimal because of infinitesimal constant ε appearing in (11).

The Lie derivatives of a vector field X^i, connection parameters G^i_{jk} and curvature tensor H^i_{jkh} are given by[2] .

$$\pounds X^i = v^j x^i_{(j)} - X^j v^i_{(j)} + \dot{\partial}_j X^i v^j_{(l)} \dot{x}^l, \tag{12}$$

$$\pounds G^i_{jk} = v^i_{(j)(k)} + v^h H^i_{hjk} + G^i_{jkh} v^h_{(l)} \dot{x}^l, \tag{13}$$

$$\begin{aligned} \pounds H^i_{jkh} \;=\; & v^l H^i_{jkh(l)} - H^l_{jkh} v^i_{(l)} + H^i_{lkh} v^l_{(j)} + H^i_{jlh} v^l_{(k)} + \\ & H^i_{jkl} v^l_{(h)} + \dot{\partial}_l H^i_{jkh} v^l_{(m)} \dot{x}^m. \end{aligned} \tag{14}$$

The process of Lie differentiation and other differentiation of any arbitrary tensor and connection coefficient G^i_{jk} are connected by

$$\dot{\partial}_l(\pounds T^i_{jk}) - \pounds(\dot{\partial}_l T^i_{jk}) = 0, \tag{15}$$

$$\pounds\{(T^i_j)_{(k)}\} - (\pounds T^i_j)_{(k)} = \pounds G^i_{kl} T^l_j - \pounds G^l_{kj} T^i_l - \dot{\partial}_h G^i_k T^h_j, \tag{16}$$

$$(\pounds\ G^i_{jh})_{(k)} - (\pounds\ G^i_{kh})_{(j)} = \pounds H^i_{hjk} + (\pounds\ G^r_{kl}) G^i_{rjh} \dot{x}^l - (\pounds\ G^r_{jl}) G^i_{rhk} \dot{x}^l. \tag{17}$$

As indicated by K. Takano[4] the covariant vector field may assume any one of following alternative forms.

$$v^i_{(l)} = 0, \tag{18}$$

$$v^i_{(l)} = c\delta^i_j, \tag{19}$$

$$v^i_{(l)} = \beta\delta^i_j. \tag{20}$$

Accordingly, the vector field v^i is respectively called as a contra vector field, concurrent vector field and special concircular vector field. Where v^i is a scalar function and c being a non zero constant.

Special Finsler Spaces

Definition 1.*In Fn, the Berwald's curvature tensor satisfies the relation*

$$H^i_{jhk(l)} = \lambda_l H^i_{jhk}, \tag{21}$$

is called recurrent Finsler space denoted by $HR{-}F_n$[5].

and

$$H^i_{jhk} \neq 0, \tag{22}$$

where λ_l is non zero scalar vector independent of the directional arguments.

Definition 2.*In F_n, the covariant derivative of Berwald's curvature tensor is zero at every point, is called a symmetric Finsler space[6].*

$$H^i_{jhk(l)} = 0. \tag{23}$$

2.Curvature inheritance and Ricci Inheritance symmetry

The curvature inheritance(CI) defined as an infinitesimal transformation with respect to which the Lie derivative of Berwald's curvature tensor H^i_{jkh} satisfies a relation of the form

$$\pounds H^i_{jkh} = \alpha H^i_{jkh}. \tag{24}$$

Similarly, we have define the curvature inheritance symmetry satisfies by relative curvature tensor of the form

$$\pounds \tilde{K}^i_{jkh} = \alpha \tilde{K}^i_{jkh}. \tag{25}$$

Where $\alpha = \alpha(x)$ is a scalar function. A subcase of CI is the well Known symmetry curvature collineation ($\pounds H^i_{jkh} = 0$) denoted by CC, when $\alpha = 0$. In the sequel we say that CI is a proper if $\alpha \neq 0$.

We also define Ricci-Inheritance(RI) Symmetry

$$\pounds H_{jk} = \alpha H_{jk}, \tag{26}$$

and for relative curvature tensor

$$\pounds \tilde{K}_{jk} = \alpha \tilde{K}_{jk}, \tag{27}$$

If α follows the invariance property, its Berwald's covariant derivative vanish.

$$\alpha_{(k)} = 0. \tag{28}$$

Let us set

$$\pounds g_{ij} = h_{ij}. \tag{29}$$

Where g_{ij} is a metric tensor defined Rund[1].

$$g_{ij}(x, \dot{x}) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x, \dot{x}). \tag{30}$$

Theorem 1. *A necessary condition for a vector v^i to defined a CI is*

$$(h_{ij})_{(h)(k)} - (h_{ij})_{(k)(h)} = \frac{\partial h_{ij}}{\partial \dot{x}^r} H^r_{hk}. \tag{31}$$

Proof. The curvature tensor satisfies the following identity

$$g_{rj}H^r_{ihk} + g_{ri}H^r_{jhk} = 0. \tag{32}$$

Taking the Lie derivative of (32) and using (24), (29) and (32), we get

$$h_{rj}H^r_{ihk} + h_{ri}H^r_{jhk} = 0. \tag{33}$$

Using (33) in commutation formula (5), we obtain(31).

Theorem 2. *A necessary condition for a vector v^i to defined a CI is (for relative curvature tensor).*

$$h_{ij;hk} - h_{ij;kh} = 0. \tag{34}$$

Proof: The relative curvature tensor satisfies the identity

$$g_{rj}\tilde{K}^r_{ihk} + g_{ri}\tilde{K}^r_{jhk} = 0. \tag{35}$$

Taking the Lie derivative of (35) and using (25), (35), we get

$$h_{rj}\tilde{K}^r_{ihk} + h_{ri}\tilde{K}^r_{jhk} = 0. \tag{36}$$

Using (36) in commutation formula (9), we obtain (34).

The infinitesimal transformation is said to be an affine motion if it satisfies the condition

$$\pounds G^i_{jh} = 0. \tag{37}$$

Applying (24) and (37) in (17), we get

$$H^i_{hjk} = 0. \tag{38}$$

Hence we have

Theorem 3. *In F_n, the necessary condition is that curvature inheriting symmetry admitted an affine motion, space is flat.*

Contracting equation (38) with respect to indices i and k, we obtain

$$H_{hj} = 0. \tag{39}$$

Corollary 1. *In F_n, the necessary and sufficient condition that Ricci curvature inheritance symmetry admitted an affine motion, then space is Ricci flat.*

Now we considered the consequence of CI characterized by the equation (24). Its successive transvection with the directional coordinates, and using (7)(g), (7)(h), and (7)(i), we obtain

$$\begin{cases} a) & \pounds H^i_{jk} = \alpha H^i_{jk}, \\ b) & \pounds H^i_j = \alpha H^i_j. \end{cases} \tag{40}$$

The partial differentiation of (40) with respect to directional coordinates, and applying equation (7)(b) and (15), we get (24).

Thus we say that, either of the conditions expressed by (24) and (40) are equivalent.

Accordingly we state

Theorem 4. *The necessary and sufficient conditions for an infinitesimal transformation to define CI.*

3. Relation between special Finsler spaces, special vector field and curvature inheritance.

We consider an infinitesimal transformation generated by vector field v^i satisfying the conditions (18), (19) and (20).

In view of (18), the equation (14) becomes

$$\pounds H^i_{jkh} = v^l H^i_{jkh(l)}. \tag{41}$$

Applying (23) in (41), we get

$$\pounds H^i_{jkh} = 0. \tag{42}$$

Hence we have

Theorem 5. *A symmetric Finsler space with contra vector field admits no CI other then CC.*

Using (18), (21) and (24) in (14), we have

$$\alpha H^i_{jkh} = v^l \lambda_l H^i_{jkh}. \tag{43}$$

Let us assume that a vector v^l is orthogonal to vector λ_l, that is

$$v^l \lambda_l = 0. \tag{44}$$

In view of (44), the equation (43) reduces to

$$\alpha H^i_{jkh} = 0. \tag{45}$$

which immediately reduces to

$$\alpha = 0. \tag{46}$$

In view of (22).
Using (46) in (24), we get

$$\pounds H^i_{jkh} = 0. \tag{47}$$

Conversely, If the above equation (46) is true, the equation (43) yields

$$v^l \lambda_l H^i_{jkh} = 0. \tag{48}$$

Since H^i_{jkh} is non zero in $HR{-}F_n$, which implies

$$v^l \lambda_l = 0, \tag{49}$$

which shows that a vector v^l is orthogonal to vector λ_l.
Accordingly we state
Theorem 6.*In $HR{-}F_n$, the necessary and sufficient condition for CI becomes CC is that the vector v^l is orthogonal to vector λ_l.*

In view of (19) the equation (14) becomes

$$\pounds H^i_{jkh} = v^l H^i_{jkh(l)} + 2cH^i_{jkh}. \tag{50}$$

Applying (23) in (50), we have

$$\pounds H^i_{jkh} = 2cH^i_{jkh}. \tag{51}$$

From equation (24) and (51), we get

$$\frac{\alpha}{2} = c. \tag{52}$$

Using (52) in (51), we obtain

$$\pounds H^i_{jkh} = \alpha H^i_{jkh}. \tag{53}$$

Hence we have
Theorem 7. *A symmetric Finsler space with concurrent vector field admits CI.*

Using equation (20) and (7)(f), in (14), we have

$$\pounds H^i_{jkh} = v^l H^i_{jkh(l)} + 2\beta H^i_{jkh}, \tag{54}$$

by virtue of (23), the equation (54) becomes

$$\pounds H^i_{jkh} = 2\beta H^i_{jkh}. \tag{55}$$

Conversely, If the above equation (55) is true, the equation (54) reduces to

$$v^l H^i_{jkh(l)} = 0. \tag{56}$$

Since, v^l is non zero vector, it implies

$$H^i_{jkh(l)} = 0. \tag{57}$$

which shows that the space is symmetric.
Thus we state
Theorem 8. *A symmetric Finsler space with special concircular vector field, the necessary and sufficient condition for the space admits CI is that the space is symmetric.*

3.CI projective motion

In this section we shall discuss the possibilities of an infinitesimal transformation to be simultaneously CI projective motion we drive an explicit expression which admitting a CI projective motion and obtain it possibilities in symmetric Finsler space.

The condition of projective motion in infinitesimal transformation of the form

$$\pounds G^i_{kj} = 2\delta^i_{(k} p_{j)} + \dot{x}^i p_{kj}, \tag{58}$$

$$\begin{cases} (a) & p_k = \dot{\partial}_k p, \\ (b) & G^i_{kj}\dot{x}^j = G^i_k. \end{cases} \tag{59}$$

where $p(x, \dot{x})$ is homogeneous scalar function of degree one in $\dot{x}^i$. The consequent equation of projective motion is given by

$$£G^i_k = \delta^i_k p + \dot{x}^i p_{k,} \tag{60}$$

(While driving (4.3), we have used (4.2)(b) and homogeneous properties of p.)

According theorem(4) the equation (40) gives a condition equivalent to that given by (24), therefore for simplicity (40) may be taken as defining equation for CI. Applying (58), (60) and (40)b in commutation formula (16) and using homogeneous properties of H^i_j, we obtain

$$\begin{aligned} £\{(H^i_j)_{(k)}\} - H^i_j(\alpha)_{(k)} - \alpha(H^i_j)_{(k)} &= H^l_j(p_l\delta^i_k + \dot{x}^i p_{kl}) \\ &\quad -(p_j H^i_k + 2p_k H^i_j + p\dot{\partial}_k H^i_j). \end{aligned} \tag{61}$$

Contracting the equation (61) with respect to indices i, j and using (7)i, we have

$$£\{(H)_{(k)}\} - H(\alpha)_{(k)} - \alpha(H)_{(k)} = -(2Hp_k + p\dot{\partial}_k H). \tag{62}$$

Transvecting the equation (62) by $\dot{x}^i$ and using homogeneous properties of p and H, we get

$$p = \frac{1}{4}\left[(\alpha)_{(k)} + \frac{\alpha(H)_{(k)}}{H} - \frac{£\{(H)_{(k)}\}}{H}\right]. \tag{63}$$

Thus we state

Theorem 9. *The Complete integral of CI projective motion in a manifold of non zero curvature is given by (63).*

For a symmetric Finsler space vanishing the covariant derivative of non zero curvature tensor the equation (63) reduces to

$$p = \frac{1}{4}(\alpha)_{(k)}, \tag{64}$$

In the view of (28) the equation (64) immediately reduces to

$$p = 0. \tag{65}$$

Thus, the projective motion under the consideration reduces to an affine motion. we, therefore, have the

Theorem 10. *There exist non- trivial CI projective motion in a symmetric Finsler manifold.*

References

[1] H. Rund, *The Differential geommetry of Finsler spaces*, Springer-Verlag, (1959).

[2] K. Yano, *The theory of Lie derivative and its applications*, North-Holland, Amsterdam, (1957).

[3] K.L. Duggal, *Curvature inheriting symmetry in Riemannian space with application of fluid space time*, J. Math . Phys. 33(9), (1992).

[4] K.Takano, *Affine motion in non-Riemannian K*-spaces I, II, III (with M. Okumara), IV, V*, Tensor, N. S., 11(1961), 137-143.

[5] R. N. Sen, *Finsler space of recurrent curvature tensors*, Tensor, N.S., 19(1968), 291-199.

[6] R. B. Misra, *A symmetric Finsler space*, Tensor, N. S., 24(1972), 346-350.

[7] K.L. Duggal and R. Sharma, *Ricci curvature inheriting symmetry of semi Riemannian manifolds*, Contemporary Mathematics 170, (1994).

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