Acta Universitatis Apulensis No. 30/2012
ISSN: 1582-5329 pp. 126-133

THE STUDY OF CARTAN SPACE WITH RANDERS METRIC

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Abstract. In this paper, we study the Cartan space with some (α, β) metrics, in particular Randers metric admitting h-metrical d-connection. Further, we show that the condition for Cartan space with Randers metric to be locally Minkowski and Conformally flat.

2000 *Mathematics Subject Classification*: 53B40, 53C60

1. Introduction

In 1993, E. Cartan originally introduced a Cartan space, which is considered as a dual of Finsler space [1]. H. Rund [4], F. Brickell [2] and others studied the relation between these two spaces. R. Miron ([10], [11]), introduced the theory of Hamilton space and proved that Cartan space is a particular case of Hamilton space. T. Igrashi ([18], [19]), introduced the notion of the (α, β)-metric in Cartan space and obtained the metric tensor and the invariants ρ and r, which characterize the special classes of Cartan spaces with (α, β)-metric. H.G. Nagaraja [5] studied the h-metrical d-connection on Cartan space with (α, β)-metric.

The concept of Randers metric was proposed by physicist G. Randers in 1941 from the stand point of general relativity [3]. Many Finslerian geometers have made efforts in investigating the geometric properties of Randers metric.

M. S. Knebelman [8] initiated the conformal theory of Finsler spaces in 1929. Several authors including S. K. Narasimhamurthy [13] discussed conformal transformations on special Finsler spaces. P. N. Pandey [9] studied the groups of conformal transformations in conformally related Finsler spaces. M. Matsumoto [6] determined the condition for conformally flatness of Randers metric. In this paper, we consider particular Cartan space with (α, β)-metric, i.e., Randers metric $K = \alpha + \beta$ admitting a h-metrical d-connection.

2. Preliminaries

Let M be a real smooth manifold and (T^*M, Π, M), its cotangent bundle. A Cartan structure on M is a function $K : T^*M \to [0, \infty)$ with the following properties:
1. K is C^∞ on $T^*M/0$ for $0 = \{(x, 0), x \in M\}$,
2. $K(x, \lambda p) = \lambda K(x, p)$ for all $\lambda > 0$,
3. The $n \times n$ matrix (g^{ij}) is positive definite at all points of $T^*M/0$,
where $g^{ij}(x, p) = \frac{1}{2}\partial^i\partial^j K^2(x, p)$. We note that infact $K(x, p) > 0$, whenever $p \neq 0$.

Definition 1 *The pair $(M, K) = C^n$ is called Cartan space.*

Example 1 *[12] Let $(\gamma_{ij}(x))$ be the matrix of the local coefficients of a Riemmannian metric on M and $\gamma^{ij}(x)$, its inverse. Then $K(x, p) = \sqrt{\gamma^{ij}(x)p_ip_j}$ gives a Cartan structure. Thus any Riemannian manifold can be regarded as a Cartan space .*

A Cartan space $C^n = (M, K)$ is said to be with (α, β)-metric if $K(x, p)$ is a function of the variable $\alpha(x, p) = (a^{ij}p_ip_j)^{\frac{1}{2}}$ and $\beta(x, p) = b^i(x)p_i$, where a^{ij} is a Riemannian metric and $b^i(x)$ is a vector field depending only on x. Clearly K must satisfy the conditions imposed on the fundamental function of a Cartan space. The fundamental tensor $g^{ij}(x, p)$ and its reciprocal $g_{ij}(x, p)$ of the Cartan space $C^n = (M, K(\alpha, \beta))$ are given by the relation

$$g^{ij} = \rho a^{ij} + \rho_0 b^i b^j + \rho_{-1}(b^i p^j + b^j p^i) + \rho_{-2} p^i p^j, \quad (1)$$

where $\rho,\ \rho_0,\ \rho_{-1},\ \rho_{-2}$ are invariants given by
$\rho = \frac{1}{2}\alpha^{-1}K_\alpha, \quad \rho_{-1} = \frac{1}{2}\alpha^{-1}K_{\alpha\beta}, \quad \rho_{-2} = \frac{1}{2}\alpha^{-2}(K_{\alpha\alpha} - \alpha^{-1}K_\alpha), \quad \rho_0 = \frac{1}{2}K_{\beta\beta}$
and

$$g_{ij} = \sigma a_{ij} + \sigma_0 b_i b_j + \sigma_{-1}(b_i p_j + b_j p_i) + \sigma_{-2} p_i p_j, \quad (2)$$

where $\sigma = \frac{1}{\rho}, \quad \tau = \sigma + \sigma_0 B^2 + \rho_{-1}\beta, \quad \sigma_0 = \frac{\rho_0}{\rho\tau}, \quad \sigma_{-1} = \frac{\rho_{-1}}{\rho\tau}, \quad \sigma_{-2} = \frac{\rho_{-2}}{\rho\tau}$
and $B^2 = b^i b_i$.
The Cartan tensor C^{ijk} is given by

$$\begin{aligned} C^{ijk} &= -\frac{1}{2}[r_{-1}b^ib^jb^k + \{\rho_{-1}a^{ij}b^k + \rho_{-2}a^{ij}p^k + r_{-2}b^ib^jp^k \\ &+ r_{-3}b^ip^jp^k + (i/j/k)\} + r_{-4}p^ip^jp^k], \end{aligned} \quad (3)$$

where $r_{-1} = \frac{1}{2}K_{\beta\beta\beta}, \quad r_{-2} = \frac{1}{2}\alpha^{-1}K_{\alpha\beta\beta}, \quad r_{-3} = \frac{1}{2}\alpha^{-2}(K_{\alpha\alpha\beta} - \alpha^{-1}K_{\alpha\beta}),$
$r_{-4} = \frac{1}{2}\alpha^{-3}\{K_{\alpha\alpha\alpha} - 3\alpha^{-1}K_{\alpha\alpha} + 3\alpha^{-2}K_\alpha\}.$
and (i/j/k) represent cyclic sum in the indices i, j, k.

Let $' : '$ denote the covariant differentiation with respect to Christoffel symbols γ^i_{jk} constructed from a_{ij}. Since $a^{ij}_{:k} = 0$ and $p_{i:k} = 0$, if $b^i_{:k} = 0$, then $g^{ij}_{:k} = 0$. Using the Christoffel symbols $H^i_{jk} = \frac{1}{2}g^{ir}(\delta_j g_{rk} + \delta_k g_{jr} - \delta_r g_{jk})$ constructed from $g_{ij}(x,p)$, we can define canonical N-connection.

$$N_{ij} = \Gamma^k_{ij}p_k - \frac{1}{2}\Gamma^k_{hr}p_k p^r \dot{\partial}^h g_{ij}, \tag{4}$$

where $\Gamma^i_{jk}(p) = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$.
We consider the canonical d-connection

$$D\Gamma = (N_{jk}, H^i_{jk}, C^{jk}_i). \tag{5}$$

The d-connection field of type $(2,1)$, C^{jk}_i is given by

$$C^{jk}_i = -\frac{1}{2}g_{ir}\dot{\partial}^r g^{jk} = g_{ir}C^{rjk}. \tag{6}$$

We denote $'|_k{}'$ be the h-covariant differentiation with respect to $D\Gamma$.

Definition 2 *[5] A d-connection $D\Gamma$ of a Cartan space C^n with (α,β)-metric is called the h-metrical d-connection if it satisfies the conditions*
(i) h-deflection $D_{ij}(= p_{i|j})$=0,
(ii) $\alpha^{ij}_{|k} = 0$,
(iii) $g^{ij}_{|k} = 0$.

3.Cartan space with Randers metric admitting h-metrical d-connection

Let $C^n = (M, K(\alpha,\beta))$ be an n-dimensional Cartan space with the metric $K = \alpha + \beta$ where $\alpha = (a^{ij}p_i p_j)^{\frac{1}{2}}$ is a Riemannian metric and $\beta = b^i(x)p_i$ is a differential 1-form. The angular metric tensor is given by

$$h^{ij} = K\left(\frac{a^{ij}}{\alpha} - \frac{P^i P^j}{\alpha^3}\right).$$

The fundamental tensor $g^{ij}(x,p)$ and its reciprocal $g_{ij}(x,p)$ of the Cartan space $C^n = (M, K(\alpha,\beta))$ are as follows

$$g^{ij} = h^{ij} + k^i k^j, \tag{7}$$

where $k^i = \dot{\partial}^i K = b^i + \frac{P^i}{\alpha}$.

$$g_{ij} = \frac{\alpha}{\alpha+\beta}\left(a_{ij} - \frac{1}{\alpha + 2\beta + B^2}\left(\alpha b_i b_j - b_i p_j - b_j p_i + \frac{\beta}{\alpha^2}p_i p_j\right)\right). \tag{8}$$

Proposition 1 *[17] The Cartan spaces with Rander's metric is C-reducible.*

The Cartan tensor is given by

$$C^{ijk} = h^{ij}A^k + h^{jk}A^i + h^{ki}A^j, \tag{9}$$

where $A^i = \frac{1}{2K}(b^i - \frac{\beta}{\alpha^2}P^i)$.

Contracting g_{jk} on both sides of the above equation, we have

$$C^i = g_{jk}C^{ijk} = (n+1)A^i.$$

By means of (9), which implies

$$C^{ijk} = \frac{1}{n+1}(h^{ij}C^k + h^{jk}C^i + h^{ki}C^j).$$

Theorem 1 *A Cartan space C^n with Randers metric admitting a h-metrical d-connection is locally flat if and only if the associated Riemannian space is locally flat.*

Proof. If the connection $D\Gamma$ is h-metrical, then $\alpha_{|h} = 0$, from which, we get

$$0 = K_{|h} = \alpha_{|h} + \beta_{|h} = \beta_{|h} \tag{10}$$

and

$$\beta_{|h} = b^i_{|h} p_i = 0, \tag{11}$$

which yields $b^i_{|h} = 0$.

Now from $a^{ij}_{|h} = 0$, we get $H^i_{jk} = \gamma^i_{jk}$. Hence we have

$$b^i_{:k} = 0, \tag{12}$$

and also the curvature tensor D^i_{hjk} of $D\Gamma$ coincides with the curvature tensor R^i_{hjk} of Riemmanian connection $R\Gamma = (\gamma^i_{jk}, \gamma^i_{jk}y_i, 0)$.

If $R^i_{hjk} = 0$, then $D^i_{hjk} = 0$.

Definition 3 *[5] A Cartan space C^n is a Berwald space if and only if $C^{ij}_{k|h} = 0$.*

Theorem 2 *A Cartan space with Randers metric admitting h-metrical d-connection is a Berwald space.*

Proof. The connection $D\Gamma$ is h-metrical, then $g^{ij}_{|h} = 0, \quad \alpha_{|h} = 0, \quad a^{ij}_{|h} = 0, \quad b^{k}_{|h} = 0, \quad p^{k}_{|h} = 0$.
Hence, from (7), (8) and (9), we have

$$C^{ij}_{k|h} = 0. \tag{13}$$

It is well known that [15] a locally Minkowski space is a Berwald space in which the curvature tensor vanishes. Hence, from the above theorem we have

Theorem 3 *A Cartan space with Randers metric $K = \alpha+\beta$ admitting h-metrical d-connection is locally Minkowski if and only if associated Riemannian space is locally flat.*

4. Conformal change of a Cartan space with Randers metric

Let $C^n = (M, K)$ be an n-dimensional Cartan space with Randers metric $K(\alpha, \beta) = \alpha + \beta$. By conformal change $\sigma : K \to \bar{K} : \bar{K}(\bar{\alpha}, \bar{\beta}) = e^{\sigma} K(\alpha, \beta)$, we have another Cartan space $\bar{C}^n = (M, \bar{K})$, where $\bar{\alpha} = e^{\sigma}\alpha$ and $\bar{\beta} = e^{\sigma}\beta$.

Under Conformal change and putting $\alpha = (a^{ij}(x)p_i p_j)^{\frac{1}{2}}$ and $\beta = b^i(x)p_i$, we have the following quantities [12] :

$$\bar{a}^{ij} = e^{2\sigma}a^{ij}, \;\; \bar{b}^i = e^{\sigma}b^i, \bar{g}^{ij} = e^{2\sigma}g^{ij}, \;\; \bar{g}_{ij} = e^{-2\sigma}g_{ij}, \;\; \bar{h}^{ij} = e^{2\sigma}h^{ij}, \bar{C}^{ijk} = e^{2\sigma}C^{ijk}, \;\; \bar{C}^i = C^i.$$

By the Proposition 1, we state that

Theorem 4 *Under conformal transformation C-reducible property preserves in Cartan space with Randers metric.*

Theorem 5 *In a Cartan space with Randers metric, there exist conformally invariant symmetric linear connection D^i_{jk}.*

Proof. The Christoffel symbols $\bar{\gamma}^i_{jk}$ constructed from $\bar{a}^{ij}$ are written as

$$\bar{\gamma}^i_{jk} = \gamma^i_{jk} + B^i_{jk}, \tag{14}$$

where $B^i_{jk} = \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma^i a_{jk}$ and $\sigma^i = a^{ij}\sigma_j$.
Taking covariant derivative of $\bar{b}^i$ with respect to $\bar{\gamma}^i_{jk}$, we get

$$\bar{b}^i_{:k} = e^{\sigma}(b^i_{:k} + 2\sigma_k b^i + b^r\sigma_r\delta^i_k - \sigma^i b^r a_{rk}). \tag{15}$$

Transvecting above by $\bar{b}^k$ and putting

$$M^i = \frac{1}{B^2}\{b^k b^i_{:k} - \frac{b^r_{:r}b^i}{n+4}\}, \tag{16}$$

we have

$$\sigma^i = \bar{M}^i - M^i, \tag{17}$$

from which, we get $\sigma_i = \bar{M}_i - M_i$. Using (14) and putting

$$D^i_{hj} = \gamma^i_{hj} + \delta^i_h M_j + \delta^i_j M_h - M^i a_{hj},$$

we have

$$\bar{D}^i_{hj} = D^i_{hj}. \tag{18}$$

D^i_{hj} is a symmetric conformally invariant linear connection on M.

Theorem 6 *In a Cartan space C^n with Randers metric admits h-metrical d-connection $M^i = 0$ and there exist a conformally invariant symetric linear connection D^i_{jk} such that $D^i_{jk} = \gamma^i_{jk}$ and its curvature tensor $D^i_{hjk} = R^i_{hjk}$.*

Proof. we denote by D^i_{hjk} the curvature tensor D^i_{jk}, we have from (18)

$$\bar{D}^i_{hjk} = D^i_{hjk}. \tag{19}$$

Since $b^i_{:k} = 0$, we have $M^i = 0$. Hence $D^i_{jk} = \gamma^i_{jk}$ and $D^i_{hjk} = R^i_{hjk}$.

Theorem 7 *The Cartan space C^n with Randers metric is conformally flat if and only if the conditions $D^i_{hjk} = 0$, (20) and (21) are satisfied.*

Proof. The Cartan space $\bar{C}^n$ with Randers metric $\bar{K}$ is locally Minkowski, then we have $\bar{R}^i_{hjk} = 0$ and $\bar{b}^j_{:k} = 0$.

From the Theorm (6), we have $\bar{M}^i = 0$ and $\bar{D}^i_{hjk} = \bar{R}^i_{hjk}$. Thus (17) yields $\sigma^i = M^i$, the covariant vector field M^i is locally gradient

$$M^i_{:k} = M^k_{:i} \tag{20}$$

and we have from (15)

$$b^i_{:k} = 2M_k b^i + b^r M_r \delta^i_k - M^i b^r a_{rk}. \tag{21}$$

Conversely, suppose that C^n has the gradient vector $M^i = \sigma^i$ satisfying (15). If we consider the conformal Cartan space $\bar{C}^n$, then from (17), we get $\bar{M}^i = 0$, which yields $\bar{D}^i_{jk} = \bar{\gamma}^i_{jk}$ and $\bar{D}^i_{hjk} = \bar{R}^i_{hjk}$.

Hence, $\bar{R}^i_{hjk} = 0$ follows from (18). Using the condition (21), we have from (15) that $\bar{b}^j_{:k} = 0$.

Theorem 8 *A Cartan space $C^n = (M, K)$ with $K = \alpha + \beta$ admitting h-metrical d-connection is conformally flat if and only if associated Riemannian space is locally flat.*

Proof. The associate Riemannian space (M, α) is locally flat ($R^i_{hjk} = 0$), then from (19) and Theorem (5), we have $\bar{D}^i_{hjk} = 0$, that is, the space C^n is conformally flat.

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