

ON WEAKLY B -IRRESOLUTE FUNCTIONS

N. RAJESH

ABSTRACT. The concept of b -open sets was introduced by Andrijevic. The aim of this paper is to introduce and characterize weakly b -irresolute functions by using b -open sets.

2000 *Mathematics Subject Classification*: 54C05.

1. INTRODUCTION

As generalization of open sets, b -open sets were introduced and studied by Andrijevic. This notions was further studied by Ekici [3, 4, 5], Park [7] and Caldas et al [2]. In this paper, we will continue the study of related functions with b -open [1] sets. We introduce and characterize the concepts of weakly b -irresolute functions and relationships between strongly b -irresolute functions and graphs are investigated. Throughout this paper, X and Y refer always topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X , $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of A and interior of A in X , respectively. A subset A of X is said to be b -open [1] if $A \subset \text{Cl}(\text{Int}(A)) \cup \text{Int}(\text{Cl}(A))$. The complement of b -open set is called b -closed. The intersection of all b -closed sets of X containing A is called the b -closure [1] of A and is denoted by $b\text{Cl}(A)$. A set A is b -closed if and only if $b\text{Cl}(A) = A$. The union of all b -open sets of X contained in A is called the b -interior of A and is denoted by $b\text{Int}(A)$. A set A is said to be b -regular [7] if it is b -open and b -closed. The family of all b -open (resp. b -closed, b -regular) sets of X is denoted by $BO(X)$ (resp. $BC(X)$, $BR(X)$). We have set $BO(X, x) = \{V \in BO(X) | x \in V\}$ for $x \in X$.

2. PRELIMINARIES

A point x of X is called a b - θ -cluster [7] points of $S \subset X$ if $bCl(U) \cap S \neq \emptyset$ for every $U \in BO(X, x)$. The set of all b - θ -cluster points of S is called the b - θ -closure of S and is denoted by $bCl_\theta(S)$. A subset S is said to be b - θ -closed if and only if $S = bCl_\theta(S)$. The complement of a b - θ -closed set is said to be b - θ -open.

Theorem 1 [7] *Let A be a subset of a topological space X . Then,*

(i) $A \in BO(X)$ if and only if $bCl(A) \in BR(X)$.

(ii) $A \in BO(X)$ if and only if $bInt(A) \in BR(X)$.

Theorem 2 [7] *For a subset A of a topological space X , the following properties hold:*

(i) If $A \in BO(X)$, then $bCl(A) = bCl_\theta(A)$,

(ii) $A \in BR(X)$ if and only if A is b - θ -open and b - θ -closed.

Definition 1 [7] *A topological space X is said to be b -regular if for each $F \in BC(X)$ and each $x \notin F$, there exist disjoint b -open sets U and V such that $x \in U$ and $F \subset V$.*

Theorem 3 [7] *For a topological space X , the following properties are equivalent:*

(i) X is b -regular;

(ii) For each $U \in BO(X)$ and each $x \in U$, there exists $V \in BO(X)$ such that $x \in V \subset bCl(V) \subset U$;

(iii) For each $U \in BO(X)$ and each $x \in U$, there exists $V \in BR(X)$ such that $x \in V \subset U$.

Definition 2 *A function $f : X \rightarrow Y$ is said to be b -irresolute [6] if $f^{-1}(V) \in BO(X)$ for every $V \in BO(Y)$.*

3. WEAKLY b -IRRESOLUTE FUNCTIONS

We have introduced the following definition

Definition 3 *A function $f : X \rightarrow Y$ is said to be weakly b -irresolute if for each $x \in X$ and each $V \in BO(Y, f(x))$, there exists $U \in BO(X, x)$ such that $f(U) \subset bCl(V)$.*

Clearly, every b -irresolute function is weakly b -irresolute but the converse is not true, in general, as shown by the following example.

Example 1 Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, Y\}$. Define a function $f : X \rightarrow Y$ such that $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then, clearly f is weakly b -irresolute but not b -irresolute.

Theorem 4 For a function $f : X \rightarrow Y$, the following properties are equivalent:

- (i) f is weakly b -irresolute;
- (ii) $f^{-1}(V) \subset b\text{Int}(f^{-1}(b\text{Cl}(V)))$ for every $V \in BO(Y)$;
- (iii) $b\text{Cl}(f^{-1}(V)) \subset f^{-1}(b\text{Cl}(V))$ for every $V \in BO(Y)$.

Proof. (i) \Rightarrow (ii): Suppose that $V \in BO(Y)$ and let $x \in f^{-1}(V)$. By (i), $f(U) \subset b\text{Cl}(V)$ for some $U \in BO(X, x)$. Therefore, we have $U \subset f^{-1}(b\text{Cl}(V))$ and $x \in U \subset b\text{Int}(f^{-1}(b\text{Cl}(V)))$. This shows that $f^{-1}(V) \subset b\text{Int}(f^{-1}(b\text{Cl}(V)))$.

(ii) \Rightarrow (iii): Suppose that $V \in BO(Y)$ and $x \notin f^{-1}(b\text{Cl}(V))$. Then $f(x) \notin b\text{Cl}(V)$. There exists $F \in BO(Y, f(x))$ such that $F \cap V = \emptyset$. Since $V \in BO(Y)$, we have $b\text{Cl}(F) \cap V = \emptyset$ and hence $b\text{Int}(f^{-1}(b\text{Cl}(F))) \cap f^{-1}(V) = \emptyset$. By (ii), we have $x \in f^{-1}(F) \subset b\text{Int}(f^{-1}(b\text{Cl}(F))) \in BO(X)$. Therefore, we obtain $x \notin b\text{Cl}(f^{-1}(V)) \subset f^{-1}(b\text{Cl}(V))$.

(iii) \Rightarrow (i): Let $x \in X$ and $V \in BO(Y, f(x))$. By Theorem 1, $b\text{Cl}(V) \in BR(Y)$ and $x \notin f^{-1}(b\text{Cl}(Y - b\text{Cl}(V)))$. Since $Y - b\text{Cl}(V) \in BO(Y)$, by (iii), we have $x \notin b\text{Cl}(f^{-1}(Y - b\text{Cl}(V)))$. Hence there exists $F \in BO(X, x)$ such that $F \cap f^{-1}(Y - b\text{Cl}(V)) = \emptyset$. Therefore, we obtain $f(F) \cap (Y - b\text{Cl}(V)) = \emptyset$ and hence $f(F) \subset b\text{Cl}(V)$. This shows that f is weakly b -irresolute.

Theorem 5 For a function $f : X \rightarrow Y$, the following properties are equivalent:

- (i) f is weakly b -irresolute;
- (ii) $b\text{Cl}(f^{-1}(B)) \subset f^{-1}(b\text{Cl}_\theta(B))$ for every subset B of Y ;
- (iii) $f(b\text{Cl}(A)) \subset b\text{Cl}_\theta(f(A))$ for every subset A of X ;
- (iv) $f^{-1}(F) \in BC(X)$ for every b - θ -closed set F of Y ;
- (v) $f^{-1}(V) \in BO(X)$ for every b - θ -open set V of Y .

Proof. (i) \Rightarrow (ii): Let B be any subset of Y and $x \notin f^{-1}(b\text{Cl}_\theta(B))$. Then $f(x) \notin b\text{Cl}_\theta(B)$ and there exists $V \in BO(Y, f(x))$ such that $b\text{Cl}(V) \cap B = \emptyset$. By (i), there exists $U \in BO(X, x)$ such that $f(U) \subset b\text{Cl}(V)$. Hence $f(U) \cap B = \emptyset$ and $U \cap f^{-1}(B) = \emptyset$. Consequently, we obtain $x \notin b\text{Cl}(f^{-1}(B))$.

(ii) \Rightarrow (iii): Let A be any subset of X . By (ii), we have $b\text{Cl}(A) \subset b\text{Cl}(f^{-1}(f(A)))$

$\subset f^{-1}(b\text{Cl}_\theta(f(A)))$ and hence $f(b\text{Cl}(A)) \subset b\text{Cl}_\theta(f(A))$.

(iii) \Rightarrow (iv): Let F be any b - θ -closed set of Y . Then, by (iii), we have $f(b\text{Cl}(f^{-1}(F))) \subset b\text{Cl}_\theta(f(f^{-1}(F))) \subset b\text{Cl}_\theta(F) = F$. Therefore, we have $b\text{Cl}(f^{-1}(F)) \subset f^{-1}(F)$ and hence $b\text{Cl}(f^{-1}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F) \in BC(X)$.

(iv) \Rightarrow (v): This is obvious.

(v) \Rightarrow (i): Let $x \in X$ and $V \in BO(Y, f(x))$. By Theorems 1 and 2, $b\text{Cl}(V)$ is b - θ -open in Y . Put $U = f^{-1}(b\text{Cl}(V))$. Then by (v), $U \in BO(X, x)$ and $f(U) \subset b\text{Cl}(V)$. Thus, f is weakly b -irresolute.

Theorem 6 *For a function $f : X \rightarrow Y$, the following properties are equivalent:*

- (i) f is weakly b -irresolute;
- (ii) for each $x \in X$ and each $V \in BO(Y, f(x))$ there exists $U \in BO(X, x)$ such that $f(b\text{Cl}(U)) \subset b\text{Cl}(V)$;
- (iii) $f^{-1}(F) \in BR(X)$ for every $F \in BR(Y)$.

Proof. **(i) \Rightarrow (ii):** Let $x \in X$ and $V \in BO(Y, f(x))$. Then $b\text{Cl}(V)$ is b - θ -open and b - θ -closed in Y , by Theorems 1 and 2. Now, put $U = f^{-1}(b\text{Cl}(V))$. Then by Theorem 5, $U \in BR(X)$ and hence $U \in BO(X, x)$. Therefore, $U = b\text{Cl}(U)$ and $f(b\text{Cl}(U)) \subset b\text{Cl}(V)$.

(ii) \Rightarrow (iii): Let $F \in BR(Y)$ and $x \in f^{-1}(F)$. Then $f(x) \in F$. By (ii), there exists $V \in BO(X, x)$ such that $f(b\text{Cl}(U)) \subset F$. Therefore, we have $x \in U \subset b\text{Cl}(U) \subset f^{-1}(F)$ and hence $f^{-1}(F) \in BO(X)$. Since $Y - F \in BR(Y)$, $f^{-1}(Y - F) = X - f^{-1}(F) \in BO(X)$. Therefore, we obtain $f^{-1}(F) \in BC(X)$ and hence $f^{-1}(F) \in BR(X)$.

(iii) \Rightarrow (i): Let $x \in X$ and $V \in BO(Y, f(x))$. By Theorem 1, $b\text{Cl}(V) \in BR(Y, f(x))$ and $f^{-1}(b\text{Cl}(V)) \in BR(X, x)$. Set $U = f^{-1}(b\text{Cl}(V))$. Then $U \in BO(X, x)$ and $f(U) \subset b\text{Cl}(V)$. This shows that f is weakly b -irresolute.

The proof of the following two theorems are similar to Theorem 5 and 6.

Theorem 7 *For a function $f : X \rightarrow Y$, the following properties are equivalent:*

- (i) f is weakly b -irresolute;
- (ii) $f^{-1}(V) \subset b\text{Int}_\theta(f^{-1}(b\text{Cl}_\theta(V)))$ for every $V \in BO(Y)$;
- (iii) $b\text{Cl}_\theta(f^{-1}(V)) \subset f^{-1}(b\text{Cl}_\theta(V))$ for every $V \in BO(Y)$.

Theorem 8 *For a function $f : X \rightarrow Y$, the following properties are equivalent:*

- (i) f is weakly b -irresolute;
- (ii) $b\text{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(b\text{Cl}_\theta(B))$ for every subset B of Y ;
- (iii) $f(b\text{Cl}_\theta(A)) \subset b\text{Cl}_\theta(f(A))$ for every subset A of X ;
- (iv) $f^{-1}(F)$ is b - θ -closed in X for every b - θ -closed set F of Y .
- (v) $f^{-1}(V)$ is b - θ -open in X for every b - θ -closed set V of Y .

Theorem 9 *Let Y be b -regular space. Then a function $f : X \rightarrow Y$ is weakly b -irresolute if and only if it is b -irresolute.*

Proof. Suppose that $f : X \rightarrow Y$ is weakly b -irresolute. Let V be any b -open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since Y is b -regular, by Theorem 3 there exists $F \in BO(Y)$ such that $f(x) \in F \subset b\text{Cl}(W) \subset V$. Since f is weakly b -irresolute, there exists $U \in BO(X, x)$ such that $f(U) \subset b\text{Cl}(F)$. Therefore, we have $x \in U \subset f^{-1}(V)$ and $f^{-1}(V) \in BO(X)$. This shows that f is b -irresolute.

Theorem 10 *A function $f : X \rightarrow Y$ is weakly b -irresolute if the graph function, defined by $g(x) = (x, f(x))$ for each $x \in X$, is weakly b -irresolute.*

Proof. Let $x \in X$ and $V \in BO(Y, f(x))$. Then $X \times V$ is a b -open subset of $X \times Y$ containing $g(x)$. Since g is weakly b -irresolute, there exists $U \in BO(X, x)$ such that $g(U) \subset b\text{Cl}(X \times V) \subset X \times b\text{Cl}(V)$. Therefore, we obtain $f(U) \subset b\text{Cl}(V)$.

Recall that a topological space X is said to be b - T_2 [7] if for each pair of distinct points $x, y \in X$, there exists $U \in BO(X, x)$ and $V \in BO(X, y)$ such that $U \cap V = \emptyset$.

Lemma 11 [7] *A topological space X is b - T_2 if and only if for each pair of distinct points $x, y \in X$, there exists $U \in BO(X, x)$ and $V \in BO(X, y)$ such that $b\text{Cl}(U) \cap b\text{Cl}(V) = \emptyset$.*

Theorem 12 *If Y is b - T_2 space and $f : X \rightarrow Y$ is a weakly b -irresolute injection, then X is b - T_2 .*

Proof. Let x, y be any distinct points of X . Since f is injective, we have $f(x) \neq f(y)$. Since Y is b - T_2 , by Lemma 11 there exists $V \in BO(Y, f(x))$ and $W \in BO(Y, f(y))$ such that $b\text{Cl}(V) \cap b\text{Cl}(W) = \emptyset$. Since f is weakly b -irresolute, there exists $G \in BO(X, x)$ and $H \in BO(X, y)$ such that $f(G) \subset b\text{Cl}(V)$ and $f(H) \subset b\text{Cl}(W)$. Therefore, we obtain $G \cap H = \emptyset$ and hence X is b - T_2 .

Definition 4 A function $f : X \rightarrow Y$ is said to have a strongly b -closed graph if for each $(x, y) \in (X \times Y) - G(f)$, there exists $U \in BO(X, x)$ and $V \in BO(Y, y)$ such that $(bCl(U) \times bCl(V)) \cap G(f) = \emptyset$.

Theorem 13 If Y is a $b-T_2$ space and $f : X \rightarrow Y$ is weakly b -irresolute, then $G(f)$ is strongly b -closed.

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and by Lemma 11, there exist $V \in BO(Y, f(x))$ and $W \in BO(Y, y)$ such that $bCl(V) \cap bCl(W) = \emptyset$. Since f is weakly b -irresolute, by Theorem 6 there exists $U \in BO(X, x)$ such that $f(bCl(U)) \subset bCl(V)$. Therefore, we obtain $f(bCl(U)) \cap bCl(W) = \emptyset$ and hence $(bCl(U) \times bCl(W)) \cap G(f) = \emptyset$. This shows that $G(f)$ is strongly b -closed in $X \times Y$.

Theorem 14 If a function $f : X \rightarrow Y$ is weakly b -irresolute, injective and $G(f)$ is strongly b -closed, then X is $b-T_2$.

Proof. Let x, y be a pair of distinct points of X . Since f is injective, $f(x) \neq f(y)$ and $(x, f(y)) \notin G(f)$. Since $G(f)$ is strongly b -closed, there exist $U \in BO(X, x)$ and $V \in BO(Y, f(y))$ such that $f(bCl(U)) \cap bCl(V) = \emptyset$. Since f is weakly b -irresolute, there exists $H \in BO(X, y)$ such that $f(H) \subset bCl(V)$. Therefore, we obtain $f(bCl(U)) \cap f(H) = \emptyset$ and hence $G \cap H = \emptyset$. This shows that X is $b-T_2$.

Recall that a topological space X is said to be b -connected [7] if it cannot be written as the union of two non-empty disjoint b -open sets.

Theorem 15 If a function $f : X \rightarrow Y$ is a weakly b -irresolute surjection and X is b -connected, then Y is b -connected.

Proof. Suppose that Y is not b -connected. Then there exists nonempty b -open sets U and V of Y such that $U \cup V = Y$ and $U \cap V = \emptyset$. Then we have $U, V \in BR(Y)$. By Theorem 6, $f^{-1}(U), f^{-1}(V) \in BR(X)$ since f is weakly b -irresolute. Moreover, we have $f^{-1}(U) \cup f^{-1}(V) = X$, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, and $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Therefore, X is not b -connected.

The following example shows that the image of a b -connected set under a weakly b -irresolute function is not necessarily b -connected.

Example 2 Let $X = \{a, b, c\}$, τ be the indiscrete topology and $\sigma = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. The function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined by $f(a) = a$ and $f(b) = f(c) = b$ is weakly b -irresolute. Moreover, X is b -connected but $f(X)$ is not b -connected.

Definition 5 A function $f : X \rightarrow Y$ is said to be almost b -irresolute if for each point $x \in X$ and each b -open set containing $f(x)$, $b\text{Cl}(f^{-1}(V)) \in \text{BO}(X, x)$.

An almost b -irresolute function need not be weakly b -irresolute, as show by the following example.

Example 3 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then the function $f : X \rightarrow Y$ defined by $f(a) = f(c) = b$ and $f(b) = c$ is weakly b -irresolute but not b -irresolute.

Theorem 16 If $f : X \rightarrow Y$ is almost b -irresolute adn $b\text{Cl}(f^{-1}(V)) \subset f^{-1}(b\text{Cl}(V))$ for each $V \in \text{BO}(Y)$, then f is weakly b -irresolute.

Proof. For any point $x \in X$ and $V \in \text{BO}(X, f(x))$, we have $b\text{Cl}(f^{-1}(V)) \subset f^{-1}(b\text{Cl}(V))$ by hypothesis. Since f is almost b -irresolute, there exists $U \in \text{BO}(X, x)$ such that $x \in U \subset b\text{Cl}(f^{-1}(V)) \subset f^{-1}(b\text{Cl}(V))$. Thus, $f(U) \subset b\text{Cl}(V)$. The con-

verse to Theorem 16 does not hold, in general, as shown by the following example.

Example 4 Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, Y\}$. Define a function $f : X \rightarrow Y$ such that $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then, clearly f is weakly b -irresolute but not b -irresolute.

Theorem 17 An almost b -irresolute function $f : X \rightarrow Y$ is weakly b -irresolute if and only if $b\text{Cl}(f^{-1}(V)) \subset f^{-1}(b\text{Cl}(V))$.

Proof. The proof follows from Theorems 4 and 16.

Definition 6 Let A be a subset of a space X . The weakly b -irresolute function from X onto a subspace A of X is called a weakly b -irresolute retraction if the restriction $f|_A$ is the identity function on A . We call such an A a weakly b -irresolute retract of X .

Lemma 18 [1] If A is b -open and U is open in a space X , then $A \cap U$ is b -open in X .

Theorem 19 Let A be a subset of a space X and $f : X \rightarrow A$ be a weakly b -irresolute retraction of X onto A . If X is T_2 , then A is b -closed in X .

Proof. Suppose A is not b -closed. Then there exists a b -limit point x of A in X such that $x \in b\text{Cl}(A)$ but $x \notin A$. Since f is weakly b -irresolute retraction, $f(x) \neq x$. Since X is T_2 , there exist disjoint open sets, say, U and V containing x and $f(x)$, respectively. Thus, $U \cap \text{Cl}(V) = \emptyset$. Also, $V \cap A$ is open in A ; hence $V \cap A \in \text{BO}(A, f(x))$. Let $W \in \text{BO}(X, x)$. Then $U \cap W \in \text{BO}(X, x)$, by Lemma 18, and hence $(U \cap W) \cap A \neq \emptyset$ because $x \in bd(A)$. Therefore, there exists a point $y \in (U \cap W \cap A)$. Since $y \in A$, $f(y) = y \in U$ and hence $f(y) \notin \text{Cl}(V)$. This shows that $f(W) \not\subset \text{Cl}(V)$. Now, $\text{Cl}(V \cap A) = \text{Cl}(V \cap A) \cap A \subset \text{Cl}(V)$. Therefore, $f(W)$ is not a subset of $\text{Cl}(V \cap A)$, which implied that $f(W)$ is not a subset of $b\text{Cl}(V \cap A)$. This contradicts the hypothesis that f is weakly b -irresolute. Thus, A is b -closed in X . In Theorem 19, X is necessary T_2 , as shown by the following example.

Example 5 Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, Y\}$. Define a function $f : X \rightarrow Y$ such that $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then, clearly f is weakly b -irresolute but not b -irresolute.

References

- [1] D. Andrijevic, *On b -open sets*, Math. Vesnik, 48(1996), 59–64.
- [2] M. Caldas, S. Jafari and T. Noiri, *On \wedge_b -sets and the associated topology τ^{\wedge_b}* , Acta. Math. Hungar., 110(4), (2006), 337-345.
- [3] E. Ekici, *On γ -US-spaces*, Indian J. Math., 47, 2-3(2005), 131-138.
- [4] E. Ekici, *On R -spaces*, Int. J. Pure. Appl. Math., 25(2)(2005), 163–172.
- [5] E. Ekici and M. Caldas, *Slightly γ -continuous functions*, Bol. Soc. Paran. Mat. (3s) V.22, 2 (2004), 63-74.
- [6] A. A. El-Abik, *A study of some types of mappings on topological spaces*, Master's Thesis, Faculty of Science, Tanta University, Tanta, Egypt (1997).
- [7] J. H. Park, *Strongly θ - b -continuous functions*, Acta Math. Hungar., 110(4)2006, 347-359.

Neelamegarajan Rajesh
 Department of Mathematics
 Rajah serfoji Govt. College
 Thanjavur 613005, Tamilnadu, India.
 email: nrajesh_topology@yahoo.co.in