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ON THE UNIVALENCE OF CERTAIN GENERAL INTEGRAL OPERATOR

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Abstract. In this paper we generalize an integral operator given by Pescar [5] and determine sufficient conditions for univalence of this new general integral operator. Several corollaries of the main results are also considered.

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1. Introduction

Let $U = \{z \in \mathbb{C} : \ |z| < 1\}$ be the unit disk of the complex plane and denote by $H(U)$ the class of holomorphic functions in U. Consider $\mathcal{A} = \{f \in H(U) : f(z) = z + a_2 z^2 + a_3 z^3 + ..., \ z \in U\}$ be the class of analytic functions in U and let S be the subclass of $\mathcal{A}$, consisting of all univalent functions f in U.

We denote by P the class of the functions h which are analytic in U, $h(0) = 1$ and Re $h(z) > 0$ for all $z \in U$.

In the present paper we consider some sufficient conditions for the following general integral operator be in the class S:

$$F(z) = \left\{ \beta \int_0^z \prod_{i=1}^{n} (g_i(u))^{\alpha} \prod_{j=1}^{p} h_j(u) u^{\beta - n\alpha - 1} du \right\}^{\frac{1}{\beta}} \tag{1}$$

for some α, β be complex numbers, $\beta \neq 0$, the functions $g_i(u) \in \mathcal{A}$, for all $i = 1, 2, ..., n$ and $h_j(u) \in P$ for all $j = 1, 2, ..., p$.

Remark 1. *It is interesting to note that the integral operator $F(z)$ defined in (1) generalizes an integral operator which was introduced and studied by V. Pescar in [5], namely*

$$B_{\alpha,\beta}(z) = \left\{\beta \int_0^z (g(u))^\alpha h(u) u^{\beta-\alpha-1} du\right\}^{\frac{1}{\beta}}, \tag{2}$$

with $\alpha,\ \beta$ be complex numbers, $\beta \neq 0,\ g \in \mathcal{A}$ and $h \in P$.

In order to prove our main results we need the following theorems:

Theorem 1. *[4] Let α be a complex number, Re $\alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3}$$

for all $z \in U$, then for any complex number β, Re $\beta \geq$ Re α, the function

$$F_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} f'(u) du\right\}^{\frac{1}{\beta}} \tag{4}$$

is in the class S.

Theorem 2. *[3]If the function $g(z)$ is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$, the following inequalities hold:*

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}\cdot g(\xi)}\right| \leq \left|\frac{\xi - z}{1-\overline{z}\cdot\xi}\right| \tag{5}$$

and

$$|g'(z)| \leq \frac{1-|g(z)|^2}{1-|z|^2}. \tag{6}$$

The equalities hold only in the case $g(z) = \dfrac{\varepsilon(z+u)}{1+\overline{u}z}$, where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 2. *[3] For $z = 0$, from inequality (5) we have*

$$\left|\frac{g(\xi)-g(0)}{1-\overline{g(0)}\cdot g(\xi)}\right| \leq |\xi| \tag{7}$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1+|g(0)|\cdot|\xi|} \tag{8}$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z|+|a|}{1+|a|\cdot|z|}, \tag{9}$$

for all $z \in U$.

Theorem 3. *[5] Let α be a complex number, Re $\alpha > 0$, M, N real positive numbers, the functions $g \in \mathcal{A}$, $g(z) = z + a_2z^2 + a_3z^3 + ...$ and $h \in P$. If*

$$\left|\frac{zg'(z)}{g(z)} - 1\right| \leq M, \ (z \in U)$$

$$\left|\frac{zh'(z)}{h(z)}\right| \leq N, \ (z \in U)$$

and

$$|\alpha| \cdot M + N \leq Re\ \alpha$$

then for every complex number β, Re $\beta \geq$ Re α, the integral operator $B_{\alpha,\beta}$ given by (2) is in the class S.

Theorem 4. *[5] Let α be a complex number, Re $\alpha > 0$, M, N real positive numbers, $M \in (0,1)$, the functions $h \in P$, $h'(0) = 0$ and $g \in \mathcal{A}$, $g(z) = z + a_2z^2 + a_3z^3 + ...$ If*

$$\left|\frac{zg'(z) - g(z)}{zg(z)}\right| < M, \ (z \in U)$$

$$\left|\frac{h'(z)}{h(z)}\right| < N, \ (z \in U)$$

and

$$\frac{M}{1-N} < |\alpha| \leq \frac{1}{\max\limits_{|z|<1}\left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} \cdot |z| \cdot \frac{|z| + |a_2|}{1 + |a_2| \cdot |z|}\right]},$$

then for every complex number β, Re $\beta \geq$ Re α, the integral operator $B_{\alpha,\beta}$ defined by (2) is in the class S.

Lemma 1. *[2](General Schwarz Lemma). Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} \cdot |z|^m \quad (z \in U_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m,$$

where θ is constant.

2. Main results

Theorem 5. *Let α be a complex number, Re $\alpha > 0$, $M_i > 0$, for all $i = 1, 2, ...n$, $N_j > 0$, for all $j = 1, 2, ...p$, the functions $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + ...$ for all $i = 1, 2, ...n$ and $h_j \in P$ for all $j = 1, 2, ...p$.*
If

$$\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \leq M_i, \text{ for all } i = 1, 2, ...n, \ (z \in U) \tag{10}$$

$$\left|\frac{zh_j'(z)}{h_j(z)}\right| \leq N_j, \text{ for all } j = 1, 2, ...p, \ (z \in U) \tag{11}$$

and

$$|\alpha| \cdot \sum_{i=1}^{n} M_i + \sum_{j=1}^{p} N_j \leq Re\ \alpha \tag{12}$$

then for every complex number β, Re $\beta \geq$ Re α, the integral operator $F(z)$ given by (1) is in S.

Proof. We observe that

$$F(z) = \left\{\beta \int_0^z \prod_{i=1}^{n} \left(\frac{g_i(u)}{u}\right)^{\alpha} \prod_{j=1}^{p} h_j(u) u^{\beta-1} du\right\}^{\frac{1}{\beta}} \tag{13}$$

Let us define the function

$$f(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_i(u)}{u}\right)^{\alpha} \prod_{j=1}^{p} h_j(u) du, \ (z \in U) \tag{14}$$

for $g_i \in \mathcal{A}$, for all $i = 1, 2, ...n$ and $h_j \in P$, for all $j = 1, 2, ...p$.
The function f is regular in U and $f(0) = f'(0) - 1 = 0$. We have

$$\frac{zf''(z)}{f'(z)} = \alpha \sum_{i=1}^{n} \left(\frac{zg_i'(z)}{g_i(z)} - 1\right) + \sum_{j=1}^{p} \frac{zh_j'(z)}{h_j(z)}, \ (z \in U). \tag{15}$$

From (10), (11) and (15) we obtain

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left(|\alpha| \cdot \sum_{i=1}^{n} M_i + \sum_{j=1}^{p} N_j\right), \ (z \in U) \tag{16}$$

and by (12), we have

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \text{ for all } z \in U. \tag{17}$$

From (14) we obtain

$$f'(z) = \prod_{i=1}^{n} \left(\frac{g_i(z)}{z} \right)^{\alpha} \prod_{j=1}^{p} h_j(z)$$

and using (17) by Theorem 1, it results that the integral operator F given by (1) is in the class S. □

Letting $p = 1$ in Theorem 5, we have

Corollary 1. *Let α be a complex number, Re $\alpha > 0$, $M_i > 0$, for all $i = 1, 2, ...n$, $N > 0$, the functions $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + ...$ for all $i = 1, 2, ...n$ and $h \in P$.*
If

$$\left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| \le M_i, \; for \; all \; i = 1, 2, ...n, \; z \in U$$

$$\left| \frac{z h'(z)}{h(z)} \right| \le N, \; (z \in U)$$

and

$$|\alpha| \cdot \sum_{i=1}^{n} M_i + N \le Re \; \alpha$$

then for every complex number β, Re $\beta \ge$ Re α, the integral operator

$$H(z) = \left\{ \beta \int_0^z \prod_{i=1}^{n} (g_i(u))^{\alpha} h(u) u^{\beta - n\alpha - 1} du \right\}^{\frac{1}{\beta}}$$

is in the class S.

Letting $n = 1$ in Theorem 5, we have

Corollary 2. *Let α be a complex number, Re $\alpha > 0$, $M > 0$, $N_j > 0$, for all $j = 1, 2, ...p$, the functions $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + a_3 z^3 + ...$ and $h_j \in P$ for all $j = 1, 2, ...p$.*
If

$$\left| \frac{z g'(z)}{g(z)} - 1 \right| \le M, \; (z \in U),$$

$$\left| \frac{z h_j'(z)}{h_j(z)} \right| \le N_j, \; for \; all \; j = 1, 2, ...p, \; (z \in U)$$

and

$$|\alpha| \cdot M + \sum_{j=1}^{p} N_j \le Re \; \alpha$$

then for every complex number β, $Re\ \beta \geq Re\ \alpha$, the integral operator

$$G(z) = \left\{ \beta \int_0^z (g(u))^\alpha \prod_{j=1}^{p} h_j(u) u^{\beta-\alpha-1} du \right\}^{\frac{1}{\beta}}$$

is in the class S.

For $M_1 = M_2 = ... = M_n = M$ and $N_1 = N_2 = ... = N_p = N$ in Theorem 5, we have

Corollary 3. *Let α be a complex number, $Re\ \alpha > 0$, $M > 0$, $N > 0$, the functions $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + ...$ for all $i = 1, 2, ...n$ and $h_j \in P$ for all $j = 1, 2, ...p$. If*

$$\left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| \leq M, \ for\ all\ i = 1, 2, ...n,\ (z \in U)$$

$$\left| \frac{z h_j'(z)}{h_j(z)} \right| \leq N, \ for\ all\ j = 1, 2, ...p,\ (z \in U)$$

and

$$|\alpha| nM + pN \leq Re\ \alpha$$

then for every complex number β, $Re\ \beta \geq Re\ \alpha$, the integral operator $F(z)$ given by (1) is in S.

Remark 3. *Letting $n = 1$, $p = 1$, $g_1 = g$ and $h_1 = h$ in Theorem 5, we obtain Theorem 3 given by V. Pescar [5].*

Theorem 6. *Let α be a complex number, $Re\ \alpha > 0$, $M_i > 0$, for all $i = 1, 2, ...n$, $\sum_{i=1}^{n} M_i \in (0, 1)$, $N_j > 0$, for all $j = 1, 2, ...p$, the functions $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + ...$ for all $i = 1, 2, ...n$ and $h_j \in P$, $h_j'(0) = 0$ for all $j = 1, 2, ...p$. If*

$$\left| \frac{z g_i'(z) - g_i(z)}{z g_i(z)} \right| < M_i, \ for\ all\ i = 1, 2, ...n,\ (z \in U) \tag{18}$$

$$\left| \frac{h_j'(z)}{h_j(z)} \right| < N_j, \ for\ all\ j = 1, 2, ...p,\ (z \in U) \tag{19}$$

and

$$\frac{\sum_{j=1}^{p} N_j}{1 - \sum_{i=1}^{n} M_i} < |\alpha| \leq \frac{1}{\max_{|z|<1} \left[\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \cdot |z| \cdot \frac{|z| + \left| \sum_{i=1}^{n} a_2^i \right|}{1 + \left| \sum_{i=1}^{n} a_2^i \right| \cdot |z|} \right]}, \tag{20}$$

then for every complex number β, Re $\beta \geq$ Re α, the integral operator $F(z)$ given by (1) is in S.

Proof. The integral operator $F(z)$ is of the form (13). We define the function

$$f(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_i(u)}{u}\right)^{\alpha} \prod_{j=1}^{p} h_j(u)du, \ (z \in U) \tag{21}$$

with $g_i \in \mathcal{A}$, for all $i = 1, 2, ...n$ and $h_j \in P$, for all $j = 1, 2, ...p$.
We consider the function

$$K(z) = \frac{1}{|z|} \cdot \frac{f''(z)}{f'(z)}, \text{ for all } z \in U. \tag{22}$$

We have

$$\frac{1}{|z|} \cdot \left|\frac{f''(z)}{f'(z)}\right| \leq \sum_{i=1}^{n} \left|\frac{zg_i'(z) - g_i(z)}{zg_i(z)}\right| + \frac{1}{|\alpha|} \sum_{j=1}^{p} \left|\frac{h_j'(z)}{h_j(z)}\right|, z \in U. \tag{23}$$

From (18), (19) and (23), we have

$$|K(z)| < \sum_{i=1}^{n} M_i + \frac{1}{|\alpha|} \sum_{j=1}^{p} N_j$$

and, using (20), we obtain $|K(z)| < 1$ for all $z \in U$.
We have $K(0) = \frac{\alpha}{|\alpha|} \sum\limits_{i=1}^{n} a_2^i$ and, using Remark 2, we get

$$|K(z)| \leq \frac{|z| + \left|\sum\limits_{i=1}^{n} a_2^i\right|}{1 + \left|\sum\limits_{i=1}^{n} a_2^i\right| \cdot |z|}, \ (z \in U). \tag{24}$$

Let us consider the function

$$Q(x) = \frac{1 - x^{2Re \ \alpha}}{Re \ \alpha} \cdot x \cdot \frac{x + \left|\sum\limits_{i=1}^{n} a_2^i\right|}{1 + \left|\sum\limits_{i=1}^{n} a_2^i\right| \cdot x}, \ (x = |z|, \ z \in U).$$

Because $Q\left(\frac{1}{2}\right) > 0$ it results that $\max\limits_{x \in [0,1]} Q(x) > 0$. Using this result and from (22) and (24), we obtain

$$\frac{1 - |z|^{2Re \ \alpha}}{Re \ \alpha} \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq |\alpha| \cdot \max_{|z|<1} \left[\frac{1 - |z|^{2Re \ \alpha}}{Re \ \alpha} \cdot |z| \cdot \frac{|z| + \left|\sum\limits_{i=1}^{n} a_2^i\right|}{1 + \left|\sum\limits_{i=1}^{n} a_2^i\right| \cdot |z|}\right] \tag{25}$$

for all $z \in U$.
From (20) and (25), we have

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} \cdot \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \ (z \in U). \tag{26}$$

So, by Theorem 1, we obtain that the integral operator $F(z)$ given by (1) is in S. □

Letting $p = 1$ in Theorem 6, we have

Corollary 4. *Let α be a complex number, $Re\ \alpha > 0$, $M_i > 0$, for all $i = 1, 2, ...n$, $\sum_{i=1}^{n} M_i \in (0, 1)$, $N > 0$, the functions $h \in P$, $h'(0) = 0$ and $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + ...$ for all $i = 1, 2, ...n$.*
If

$$\left| \frac{zg_i'(z) - g_i(z)}{zg_i(z)} \right| < M_i, \ for\ all\ i = 1, 2, ...n, \ (z \in U)$$

$$\left| \frac{h'(z)}{h(z)} \right| < N, \ (z \in U)$$

and

$$\frac{N}{1 - \sum_{i=1}^{n} M_i} < |\alpha| \leq \frac{1}{\max_{|z|<1} \left[\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} \cdot |z| \cdot \frac{|z| + \left| \sum_{i=1}^{n} a_2^i \right|}{1 + \left| \sum_{i=1}^{n} a_2^i \right| \cdot |z|} \right]},$$

then for every complex number β, $Re\ \beta \geq Re\ \alpha$, the integral operator

$$H(z) = \left\{ \beta \int_0^z \prod_{i=1}^{n} (g_i(u))^{\alpha} h(u) u^{\beta - n\alpha - 1} du \right\}^{\frac{1}{\beta}}$$

is in the class S.

Letting $n = 1$ in Theorem 6, we have

Corollary 5. *Let α be a complex number, $Re\ \alpha > 0$, $M \in (0, 1)$, $N_j > 0$, for all $j = 1, 2, ...p$, the functions $g \in \mathcal{A}$, $g(z) = z + a_2 z^2 + a_3 z^3 + ...$ and $h_j \in P$, $h_j'(0) = 0$ for all $j = 1, 2, ...p$.*
If

$$\left| \frac{zg'(z) - g(z)}{zg(z)} \right| < M, \ (z \in U)$$

$$\left|\frac{h_j'(z)}{h_j(z)}\right| < N_j, \; for \; all \; j = 1, 2, ...p, \; (z \in U)$$

and

$$\frac{\sum\limits_{j=1}^{p} N_j}{1 - M} < |\alpha| \leq \frac{1}{\max\limits_{|z|<1}\left[\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \cdot |z| \cdot \frac{|z| + |a_2|}{1 + |a_2| \cdot |z|}\right]},$$

then for every complex number β, $Re\ \beta \geq Re\ \alpha$, the integral operator

$$G(z) = \left\{\beta \int_0^z (g(u))^{\alpha} \prod_{j=1}^{p} h_j(u) u^{\beta - \alpha - 1} du\right\}^{\frac{1}{\beta}}$$

is in the class S.

For $M_1 = M_2 = ... = M_n = M$ and $N_1 = N_2 = ... = N_p = N$ in Theorem 6, we have

Corollary 6. *Let α be a complex number, $Re\ \alpha > 0$, $M > 0$, $nM \in (0, 1)$, $N > 0$, the functions $g_i \in \mathcal{A}$, $g_i(z) = z + a_2^i z^2 + a_3^i z^3 + ...$ for all $i = 1, 2, ...n$ and $h_j \in P$, $h_j'(0) = 0$ for all $j = 1, 2, ...p$.*
If

$$\left|\frac{z g_i'(z) - g_i(z)}{z g_i(z)}\right| < M, \; for \; all \; i = 1, 2, ...n, \; (z \in U)$$

$$\left|\frac{h_j'(z)}{h_j(z)}\right| < N, \; for \; all \; j = 1, 2, ...p, \; (z \in U)$$

and

$$\frac{pN}{1 - nM} < |\alpha| \leq \frac{1}{\max\limits_{|z|<1}\left[\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \cdot |z| \cdot \frac{|z| + \left|\sum\limits_{i=1}^{n} a_2^i\right|}{1 + \left|\sum\limits_{i=1}^{n} a_2^i\right| \cdot |z|}\right]},$$

then for every complex number β, $Re\ \beta \geq Re\ \alpha$, the integral operator $F(z)$ given by (1) is in S.

Remark 4. *Letting $n = 1$, $p = 1$, $g_1 = g$ and $h_1 = h$ in Theorem 6, we obtain Theorem 4 given by V. Pescar [5].*

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