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SOME FIXED POINT THEOREMS OF PREŠIĆ - ĆIRIĆ TYPE

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Abstract. In this paper, we first prove a fixed point theorem for mappings in complete metric spaces satisfying Prešić - Ćirić type which is a generalization of the result of Ćirić and Prešić [L. B. Ćirić and S. B. Prešić, On Prešić type generalization of the Banach contraction mapping principle, Acta. Math. Univ. Comenian. LXXVI (2) (2007) 143-147]. Then we present this result in the context of ordered metric spaces by using monotone non-decreasing mappings. We also support our results by some examples.

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1. Introduction and preliminaries

Difference equations play a prominent role in economics, biology, ecology, genetics, psychology, sociology, probability theory and other disciplines. Recently, nonlinear difference equations have been studied by many authors (see for example, [2], [3], [5], [6], [14], [16]). Some known difference equations can be found, for example, in [14], [16] and references therein:

- The flour beetle population model:

$$x_{n+3} = ax_{n+2} + bx_n e^{-(cx_{n+2}+dx_n)}, \quad n \in \mathbf{N}$$

  where $a, b, c, d \geq 0$ and $c + d > 0$

- The generalized Beddington-Holt stock recruitment model:

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}, \quad x_0, x_1 > 0, \quad n \in \mathbf{N}$$

  where $a \in (0, 1)$, $b \in \mathbf{R}_+^*$ and $c, d \in \mathbf{R}_+$ with $c + d > 0$.

- The delay model of a perennial grass:

$$x_{n+1} = ax_n + (b + cx_{n-1})\, e^{x_n}, \quad n \in \mathbf{N}$$

where $a, c \in (0,1)$ and $b \in \mathbf{R}_+$.

These suggest considering the k-th order nonlinear difference equation:

$$x_{n+k} = f(x_n, x_{n+1}, ..., x_{n+k-1}), \quad n \in \mathbf{N}, \tag{1}$$

with the initial values $x_0, x_1, ..., x_k \in X$, where (X, d) is a metric space, $k \in \mathbf{N}^*$ and $f : X^k \to X$.

The equation (1) can be studied by means of a fixed point theory in view of the fact that $x^* \in X$ is a solution of (1) if and only if x^* is a fixed point of f, that is,

$$x^* = f(x^*, x^*, ..., x^*)$$

One of the most important results on this direction has been obtained by S. Prešić in [11] which is a generalization of Banach contraction mapping principle:

Theorem 1.1. ([11]) *Let (X, d) be a complete metric space, k a positive integer, $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbf{R}_+$, $\sum_{i=1}^{k} \alpha_i = \alpha < 1$ and $f : X^k \to X$ a mapping satisfying*

$$d\left(f(x_0, x_1, ..., x_{k-1}), f(x_1, x_2, ..., x_k)\right) \leq \alpha_1 d(x_0, x_1) + \alpha_2 d(x_1, x_2) + ... + \alpha_k d(x_{k-1}, x_k)$$

for all $x_0, x_1, ..., x_k \in X$.

Then:

1) f has a unique fixed point $x^* \in X$.

2) the sequence $\{x_n\}_{n \geq 0}$ defined by

$$x_{n+k} = f\left(x_n, x_{n+1}, ..., x_{n+k-1}\right), \quad n \in \mathbf{N} \tag{2}$$

converges to x^* for any $x_0, x_1, ..., x_{k-1} \in X$ and

$$\lim x_n = f\left(\lim x_n, \lim x_n, ..., \lim x_n\right).$$

Afterward, some generalizations of Theorem 1.1 were established ([4], [11], [13] and references therein). An important generalization result was obtained by Ćirić and Prešić in [4]:

Theorem 1.2. ([4]) *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ a mapping satisfying the following contractive type condition*

$$d\left(f(x_1, x_2, ..., x_k), f(x_2, x_3, ..., x_{k+1})\right) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} \tag{3}$$

where $k \in (0,1)$ *is constant and* $x_1, x_2, ..., x_{k+1}$ *are arbitrary elements in* X*. Then there exists a point* x *in* X *such that* $f(x, x, ..., x) = x$*. Moreover, if* $x_1, x_2, ..., x_{k+1}$ *are arbitrary elements in* X *and for* $n \in \mathbf{N}$

$$x_{n+k} = f(x_n, x_{n+1}, ..., x_{n+k-1})$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ *is convergent and*

$$\lim x_n = f(\lim x_n, \lim x_n, ..., \lim x_n)$$

If in addition we suppose that on diagonal $\Delta \in X^k$,

$$d(f(u, u, ..., u), f(v, v, ..., v)) < d(u, v)$$

holds for all $u, v \in X$*, with* $u \neq v$*, then* x *is the unique fixed point of* f *in* X *with* $f(x, x, ..., x) = x$*.*

In this paper, we first prove a fixed point theorem for mappings satifying nonlinear contraction of Prešić - Ćirić type in complete metric spaces which is a generalization of Theorem 1.2. Then we present this result in the context of ordered metric spaces by using monotone non-decreasing mapping.

2. Main results

Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(i) φ is continuous and non-decreasing,

(ii) $\sum_{i=l}^{\infty} \varphi^i(t) < \infty$ for all $t \in (0, \infty)$.

Lemma 2.1. ([8]) *Suppose that* $\varphi : [0, \infty) \to [0, \infty)$ *is non-decreasing. Then for every* $t > 0$*,* $\lim_{n \to \infty} \varphi^n(t) = 0$ *implies* $\varphi(t) < t$*.*

The property (ii) of φ implies $\lim_{n \to \infty} \varphi^n(t) = 0$ for every $t > 0$. Therefore, by Lemma 2.1, $\varphi \in \Phi$ then $\varphi(t) < t$ for every $t > 0$.

2.1. Fixed point theorem of Prešić – Ćirić type

In this section, we prove a fixed point theorem which is a generalization of Theorem 1.2.

Theorem 2.2 *Let* (X, d) *be a complete metric space,* k *a positive integer and mapping* $f : X^k \to X$ *. Suppose that there exists* $\varphi \in \Phi$ *such that*

$$d(f(x_1, x_2, ..., x_k), f(x_2, x_3, ..., x_{k+1})) \leq \varphi(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}) \qquad (4)$$

for all $x_1, x_2, ..., x_{k+1} \in X$. *Then there exists a point* x *in* X *such that* $f(x, x, ..., x) = x$. *Moreover, if* $x_1, x_2, ..., x_{k+1}$ *are arbitrary elements in* X *and for* $n \in \mathbf{N}$

$$x_{n+k} = f(x_n, x_{n+1}, ..., x_{n+k-1})$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ *is convergent and*

$$\lim x_n = f(\lim x_n, \lim x_n, ..., \lim x_n)$$

If in addition we suppose that on diagonal $\Delta \in X^k$,

$$d(f(u, u, ..., u), f(v, v, ..., v)) < d(u, v) \tag{5}$$

holds for all $u, v \in X$, *with* $u \neq v$, *then* x *is the unique fixed point of* f *in* X *with* $f(x, x, ..., x) = x$.

Proof. Let $x_1, x_2, ..., x_k$ be k arbitrary points in X. We define the sequence $\{x_n\}$ as follows

$$x_{n+k} = f(x_n, x_{n+1}, ..., x_{n+k-1}), \quad n = 1, 2, ...$$

Set $\theta = \max\{d(x_1, x_2), d(x_2, x_3), ..., d(x_k, x_{k+1})\}$. If $x_1 = x_2 = ... = x_k = x_{k+1} = x$ then x is a fixed point of f.Thus, we may assume that $x_1, x_2, ..., x_k, x_{k+1}$ are not all equal, that is, $\theta > 0$.
By the assumptions, we have the following estimations:

$$\begin{aligned} d(x_{k+1}, x_{k+2}) &= d(f(x_1, x_2, ..., x_k), f(x_2, x_3, ..., x_{k+1})) \\ &\leq \varphi(\max\{d(x_1, x_2), d(x_2, x_3), ..., d(x_k, x_{k+1})\}) \\ &\leq \varphi(\theta) < \theta \end{aligned}$$

$$\begin{aligned} d(x_{k+2}, x_{k+3}) &= d(f(x_2, x_3, ..., x_{k+1}), f(x_3, x_4, ..., x_{k+2})) \\ &\leq \varphi(\max\{d(x_2, x_3), d(x_3, x_4), ..., d(x_{k+1}, x_{k+2})\}) \\ &\leq \varphi(\max\{\theta, \varphi(\theta)\}) = \varphi(\theta) < \theta \end{aligned}$$

$$\cdots$$

$$\begin{aligned} d(x_{2k}, x_{2k+1}) &= d(f(x_k, x_{k+1}, ..., x_{2k-1}), f(x_{k+1}, x_{k+2}, ..., x_{2k})) \\ &\leq \varphi(\max\{d(x_k, x_{k+1}), d(x_{k+1}, x_{k+2}), ..., d(x_{2k-1}, x_{2k})\}) \\ &\leq \varphi(\max\{\theta, \varphi(\theta), ..., \varphi(\theta)\}) = \varphi(\theta) < \theta \end{aligned}$$

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d\left(f(x_{k+1}, x_{k+2}, ..., x_{2k}), f(x_{k+2}, x_{k+3}, ..., x_{2k+1})\right) \\ &\leq \varphi\left(\max\{d(x_{k+1}, x_{k+2}), d(x_{k+2}, x_{k+3}), ..., d(x_{2k}, x_{2k+1})\}\right) \\ &\leq \varphi\left(\max\{\varphi(\theta), \varphi(\theta), ..., \varphi(\theta)\}\right) = \varphi^2(\theta) < \varphi(\theta) \end{aligned}$$

and so on

$$d(x_{nk+1}, x_{nk+2}) \leq \varphi^n(\theta), \quad n \geq 1$$

or

$$d(x_{n+1}, x_{n+2}) \leq \varphi^{\left[\frac{n}{k}\right]}(\theta), \quad n \geq k \tag{6}$$

By the property (ii) of φ, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = 0 \tag{7}$$

For any $n, p \in \mathbf{N}, n > k$, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+p-1}, x_{n+p}) \\ &\leq \varphi^{\left[\frac{n-1}{k}\right]}(\theta) + \varphi^{\left[\frac{n}{k}\right]}(\theta) + ... + \varphi^{\left[\frac{n+p-2}{k}\right]}(\theta) \end{aligned} \tag{8}$$

Set

$$l = \left[\frac{n-1}{k}\right] \quad and\ m = \left[\frac{n+p-2}{k}\right]$$

then $l \leq m$. From (8), we have

$$\begin{aligned} d(x_n, x_{n+p}) \leq\ & \underbrace{\varphi^l(\theta) + \varphi^l(\theta) + ...\varphi^l(\theta)}_{k\ times} \\ & + \underbrace{\varphi^{l+1}(\theta) + \varphi^{l+1}(\theta) + ...\varphi^{l+1}(\theta)}_{k\ times} \\ & + ... + \underbrace{\varphi^m(\theta) + \varphi^m(\theta) + ...\varphi^m(\theta)}_{k\ times} \end{aligned}$$

so

$$d(x_n, x_{n+p}) \leq k \sum_{i=l}^{m} \varphi^i(\theta) \tag{9}$$

By the property (ii) of φ, we have

$$\lim_{l\to\infty} \sum_{i=l}^{\infty} \varphi^i(t) = 0$$

and, in view of (9), we have $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. This means that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \tag{10}$$

We have

$$\begin{aligned} d(x_{n+k}, f(x,x,...,x)) &= d(f(x_n, x_{n+1}, ..., x_{n+k-1}), f(x,x,...,x)) \\ &\le d(f(x_n, x_{n+1}, ..., x_{n+k-1}), f(x_{n+1}, ..., x_{n+k-1}, x)) \\ &\quad + d(f(x_{n+1}, ..., x_{n+k-1}, x), f(x_{n+2}, ..., x_{n+k-1}, x, x)) \\ &\quad + ... + d(f(x_{n+k-1}, x, ..., x), f(x,x,...,x)) \end{aligned}$$

Therefore, by (4), we have

$$\begin{aligned} d(x_{n+k}, f(x,x,...,x)) &\le \varphi\left(\max\{d(x_n, x_{n+1}), ..., d(x_{n+k-2}, x_{n+k-1}), d(x_{n+k-1}, x)\}\right) \\ &\quad + \varphi\left(\max\{d(x_{n+1}, x_{n+2}), ..., d(x_{n+k-1}, x), d(x,x)\}\right) \\ &\quad + ... + \varphi\left(\max\{d(x_{n+k-1}, x), d(x,x), ..., d(x,x)\}\right) \end{aligned}$$

Taking $n \to \infty$ and using (7), (10) and the property of φ, we have $d\left(x, f(x,x,...,x)\right) \le 0$, i.e.,

$$d\left(x, f(x,x,...,x)\right) = 0$$

That implies $x = f(x,x,..,x)$, i.e., x is a fixed point of f.

Let us assume that there exists $y \in X$ such that $y = f(y,y,...,y)$. Suppose that $y \ne x$, using (5) , we have

$$d(x,y) = d(f(x,x,...,x), f(y,y,...,y)) < d(x,y)$$

which is a contraction. Thus, $x = y$, i.e., x is the unique fixed point of f.

Remark 2.3. *In Theorem 2.2, taking $\varphi(t) = \lambda t$ for all $t \in [0,\infty)$ with $\lambda \in (0,1)$ we get the result of Ćirić and Prešić (Theorem 1.2)*

2.2. Fixed point theorem of Prešić – Ćirić type in partially ordered metric spaces

In this section, we extend Theorem 2.2 and prove a fixed point theorem for monotone nondecreasing mappings in the context of ordered metric spaces.

Let $(X, \preceq)$ be a partially ordered set. Consider on X^k the following partial order: for $(x_1, x_2, ..., x_k)$, $(y_1, y_2, ..., y_k)$ in X^k

$$(x_1, x_2, ..., x_k) \sqsubseteq (y_1, y_2, ..., y_k) \quad \Leftrightarrow \quad x_1 \preceq y_1,\, x_2 \preceq y_2, ..., x_k \preceq y_k$$

Definition 2.4. *Let $(X, \preceq)$ be a partially ordered set and $f : X^k \to X$. f is said to be monotone non-decreasing if for all $(x_1, x_2, ..., x_k)$, $(y_1, y_2, ..., y_k)$ in X^k*

$$(x_1, x_2, ..., x_k) \sqsubseteq (y_1, y_2, ..., y_k) \;\Rightarrow\; f(x_1, x_2, ..., x_k) \preceq f(y_1, y_2, ..., y_k)$$

Theorem 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) be a complete metric space, k is a positive integer and the mapping $f : X^k \to X$. Suppose that there exists $\varphi \in \Phi$ such that*

$$d\left(f(y_1, y_2, ..., y_k), f(y_2, y_3, ..., y_{k+1})\right) \le \varphi\left(\max\{d(y_i, y_{i+1}) : 1 \le i \le k\}\right) \tag{11}$$

for all $y_1, y_2, ..., y_{k+1} \in X$ and $y_1 \preceq y_2 \preceq ... \preceq y_{k+1}$.
Suppose either

(a) f is continuous or

(b) X has the property: if $\{x_n\}$ is a monotone non-decreasing sequence, $x_n \to x$ then $x_n \preceq x$ for all n.

If there exist k elements $x_1, x_2, ..., x_k \in X$ such that

$$x_1 \preceq x_2 \preceq ... \preceq x_k \text{ and } x_k \preceq f(x_1, x_2, ..., x_k)$$

Then there exists a point x in X such that $f(x, x, ..., x) = x$.
If in addition we suppose that on diagonal $\Delta \in X^k$,

$$d\left(f(u, u, ..., u), f(v, v, ..., v)\right) < d(u, v)$$

holds for all $u, v \in X$, with $u \neq v$, then x is the unique fixed point of f in X with $f(x, x, ..., x) = x$.

Proof. Let $x_1, x_2, ..., x_k$ be k points in X such that

$$x_1 \preceq x_2 \preceq ... \preceq x_k \text{ and } x_k \preceq f(x_1, x_2, ..., x_k)$$

Denote

$$x_{k+1} = f(x_1, x_2, ..., x_k) \succeq x_k$$

$$x_{k+2} = f(x_2, x_3, ..., x_{k+1}) \succeq f(x_1, x_2, ..., x_k) = x_{k+1}$$

Continuing this process, we obtain the sequence $\{x_n\}$ with

$$x_{n+k} = f(x_n, x_{n+1}, ..., x_{n+k-1}), \quad n = 1, 2, ...$$

and

$$x_1 \preceq x_2 \preceq ... \preceq x_n \preceq ... \tag{12}$$

Set $\theta = \max\{d(x_1, x_2), d(x_2, x_3), ..., d(x_k, x_{k+1})\}$. If $x_1 = x_2 = ... = x_k = x_{k+1} = x$ then x is a fixed point of f. Thus, we may assume that $x_1, x_2, ..., x_k, x_{k+1}$ are not all equal, that is, $\theta > 0$.
From (12) and (11), we have the following estimations:

$$\begin{aligned} d(x_{k+1}, x_{k+2}) &= d\left(f(x_1, x_2, ..., x_k), f(x_2, x_3, ..., x_{k+1})\right) \\ &\leq \varphi\left(\max\{d(x_1, x_2), d(x_2, x_3), ..., d(x_k, x_{k+1})\}\right) \\ &\leq \varphi\left(\theta\right) < \theta \end{aligned}$$

$$\begin{aligned} d(x_{k+2}, x_{k+3}) &= d\left(f(x_2, x_3, ..., x_{k+1}), f(x_3, x_4, ..., x_{k+2})\right) \\ &\leq \varphi\left(\max\{d(x_2, x_3), d(x_3, x_4), ..., d(x_{k+1}, x_{k+2})\}\right) \\ &\leq \varphi\left(\max\{\theta, \varphi(\theta)\}\right) = \varphi\left(\theta\right) < \theta \end{aligned}$$

$$\cdots$$

$$\begin{aligned} d(x_{2k}, x_{2k+1}) &= d\left(f(x_k, x_{k+1}, ..., x_{2k-1}), f(x_{k+1}, x_{k+2}, ..., x_{2k})\right) \\ &\leq \varphi\left(\max\{d(x_k, x_{k+1}), d(x_{k+1}, x_{k+2}), ..., d(x_{2k-1}, x_{2k})\}\right) \\ &\leq \varphi\left(\max\{\theta, \varphi(\theta), ..., \varphi(\theta)\}\right) = \varphi\left(\theta\right) < \theta \end{aligned}$$

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d\left(f(x_{k+1}, x_{k+2}, ..., x_{2k}), f(x_{k+2}, x_{k+3}, ..., x_{2k+1})\right) \\ &\leq \varphi\left(\max\{d(x_{k+1}, x_{k+2}), d(x_{k+2}, x_{k+3}), ..., d(x_{2k}, x_{2k+1})\}\right) \\ &\leq \varphi\left(\max\{\varphi(\theta), \varphi(\theta), ..., \varphi(\theta)\}\right) = \varphi^2\left(\theta\right) < \varphi\left(\theta\right) \end{aligned}$$

and so on

$$d(x_{nk+1}, x_{nk+2}) \leq \varphi^n(\theta), \quad n \geq 1$$

or

$$d(x_{n+1}, x_{n+2}) \leq \varphi^{\left[\frac{n}{k}\right]}(\theta), \quad n \geq k \tag{13}$$

By the property (ii) of φ, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = 0 \tag{14}$$

For any $n, p \in \mathbf{N}, n > k$, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+p-1}, x_{n+p}) \\ &\leq \varphi^{\left[\frac{n-1}{k}\right]}(\theta) + \varphi^{\left[\frac{n}{k}\right]}(\theta) + ... + \varphi^{\left[\frac{n+p-2}{k}\right]}(\theta) \end{aligned} \tag{15}$$

Set

$$l = \left[\frac{n-1}{k}\right] \quad and\ m = \left[\frac{n+p-2}{k}\right]$$

then $l \leq m$. From (15), we have

$$\begin{aligned} d(x_n, x_{n+p}) \quad \leq \quad & \underbrace{\varphi^l(\theta) + \varphi^l(\theta) + ...\varphi^l(\theta)}_{k\ times} \\ & + \underbrace{\varphi^{l+1}(\theta) + \varphi^{l+1}(\theta) + ...\varphi^{l+1}(\theta)}_{k\ times} \\ & +... + \underbrace{\varphi^{m}(\theta) + \varphi^{m}(\theta) + ...\varphi^{m}(\theta)}_{k\ times} \end{aligned}$$

so

$$d(x_n, x_{n+p}) \leq k \sum_{i=l}^{m} \varphi^i(\theta) \tag{16}$$

By the property (ii) of φ, we have

$$\lim_{l\to\infty} \sum_{i=l}^{\infty} \varphi^i(t) = 0$$

and, in view of (16), we have $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. This means that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \tag{17}$$

Now, suppose that the assumption (a) holds. We have

$$\begin{aligned} x = \lim_{n\to\infty} x_{n+k} &= \lim_{n\to\infty} f(x_n, x_{n+1}, ..., x_{n+k-1}) \\ &= f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_{n+1}, ..., \lim_{n\to\infty} x_{n+k-1}) = f(x, x, ..., x) \end{aligned}$$

Finally, suppose that the assumption (b) holds. Then $x_n \preceq x$ for all n (since $x_n \to x$ as $n \to \infty$).
By (11), we have

$$\begin{aligned} d(x_{n+k}, f(x, x, ..., x)) &= d(f(x_n, x_{n+1}, ..., x_{n+k-1}), f(x, x, ..., x)) \\ &\leq d(f(x_n, x_{n+1}, ..., x_{n+k-1}), f(x_{n+1}, ..., x_{n+k-1}, x)) \\ &\quad + d(f(x_{n+1}, ..., x_{n+k-1}, x), f(x_{n+2}, ..., x_{n+k-1}, x, x)) \\ &\quad +... + d(f(x_{n+k-1}, x, ..., x), f(x, x, ..., x)) \end{aligned} \tag{18}$$

Therefore, by (11) and (12), we have

$$\begin{aligned} d(x_{n+k}, f(x,x,...,x)) \leq\ & \varphi\left(\max\{d(x_n,x_{n+1}),...,d(x_{n+k-2},x_{n+k-1}),d(x_{n+k-1},x)\}\right) \\ & +\varphi\left(\max\{d(x_{n+1},x_{n+2}),...,d(x_{n+k-1},x),d(x,x)\}\right) \\ & +...+\varphi\left(\max\{d(x_{n+k-1},x),d(x,x),...,d(x,x)\}\right) \end{aligned}$$

Taking $n \to \infty$ and using (14), (17) and the property of φ, we have $d\left(x, f(x,x,...,x)\right) \leq 0$, i.e.,

$$d\left(x, f(x,x,...,x)\right) = 0$$

That implies $x = f(x,x,..,x)$, i.e., x is a fixed point of f.
The uniqueness of the fixed point x is shown as in the proof of Theorem 2.2.

Corollary 2.6. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) be a complete metric space, k is a positive integer and the mapping $f: X^k \to X$. Suppose that there exists $\lambda \in (0,1)$ such that*

$$d\left(f(y_1,y_2,...,y_k), f(y_2,y_3,...,y_{k+1})\right) \leq \lambda \max\{d(y_i,y_{i+1}) : 1 \leq i \leq k\} \tag{19}$$

for all $y_1,y_2,...,y_{k+1} \in X$ and $y_1 \preceq y_2 \preceq ... \preceq y_{k+1}$.
Suppose either
(a) f is continuous or
(b) X has the property: if $\{x_n\}$ is a monotone non-decreasing sequence, $x_n \to x$ then $x_n \preceq x$ for all n.
If there exist k elements $x_1,x_2,...,x_k \in X$ such that

$$x_1 \preceq x_2 \preceq ... \preceq x_k \text{ and } x_k \preceq f(x_1,x_2,...,x_k)$$

Then there exists a point x in X such that $f\left(x,x,...,x\right) = x$.
If in addition we suppose that on diagonal $\Delta \in X^k$,

$$d\left(f(u,u,...,u), f(v,v,...,v)\right) < d\left(u,v\right)$$

holds for all $u,v \in X$, with $u \neq v$, then x is the unique fixed point of f in X with $f\left(x,x,...,x\right) = x$.

Proof. In Theorem 2.5, taking $\varphi(t) = \lambda t$ for all $t \in [0,\infty)$, we obtain Corollary 2.6.

Example 2.7. Let $X = \{0,1,2\}$ with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a complete metric space. Consider on X the partial order:

$$x,y \in X, \quad x \preceq y \Leftrightarrow x,y \in \{0,1\} \text{ and } x \leq y$$

where $\leq$ be the usual order.

Then X has the property: if $\{x_n\}$ is a monotone non-decreasing sequence, $x_n \to x$ then $x_n \preceq x$ for all n.

Define $f : X^2 \to X$ as follows:

$$f(0,0) = f(0,1) = f(1,1) = f(1,0) = f(2,2) = 0$$

$$f(0,2) = f(2,1) = 1, \; f(1,2) = f(2,0) = 2$$

Obviously, f is monotone non-decreasing. Let $\varphi : [0,\infty) \to [0,\infty)$ be given by $\varphi = t/2$ for all $t \in [0,\infty)$.

If $y_1, y_2, y_3 \in X$ with $y_1 \preceq y_2 \preceq y_3$, then $y_1 = y_2 = y_3 = 0$ or $y_1 = y_2 = y_3 = 1$ or $y_1 = y_2 = 0, y_3 = 1$ or $y_1 = 0, y_2 = y_3 = 1$.
In all cases, we have $d(f(y_1,y_2), f(y_2,y_3)) = 0$, so

$$d(f(y_1,y_2), f(y_2,y_3)) \leq \varphi\left(\max\{d(y_1,y_2), d(y_2,y_3)\}\right)$$

Also, $d(f(0,0), f(1,1)) = 0 < 1 = d(0,1)$, $d(f(0,0), f(2,2,)) = 0 < 2 = d(0,2)$ and $d(f(1,1), f(2,2)) = 0 < 1 = d(1,1)$.
Therefore, all the conditions of Theorem 2.5 are satisfied. Applying Theorem 2.5 we can conclude that f has a unique fixed point in X. In fact, 0 is the unique fixed point of f.
However, the condition (4) does not hold when $x_1 = x_2 = 1, x_3 = 2$. In fact,

$$\varphi\left(\max\{d(1,1), d(1,2)\}\right) = \varphi(1) < 1 < 2 = d(f(1,1), f(1,2)).$$

for every $\varphi \in \Phi$.
Therefore, we can not apply this example to Theorem 2.2.

Example 2.8. Let $X = \mathbf{R}$ with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. Consider on X the usual partial order. Then (X,d) is complete and X has property: if $\{x_n\}$ is a monotone non-decreasing sequence, $x_n \to x$ then $x_n \preceq x$ for all n.
Let $f : X^2 \to X$ be given by

$$f(x,y) = \frac{x-y}{4}, \; \textit{for all} x, y \in X$$

Clearly, 0 is the unique fixed point of f. However, f is not monotone non-decreasing, so we can not apply Theorem 2.5. For all $x, y, z \in X$, we have

$$d(f(x,y), f(y,z)) = \left|\frac{x-y}{4} - \frac{y-z}{4}\right| = \left|\frac{x-y}{4} + \frac{z-y}{4}\right| \leq \frac{1}{2}\max\{d(x,y), d(y,z)\}.$$

Thus, f satisfies (11) with $\varphi(t) = t/2$ for all $t \geq 0$.
Obviously, for all $x \neq y$, $d(f(x,x), f(y,y)) < d(x,y)$. Therefore, all the conditions of Theorem 2.2 are satisfied. Applying Theorem 2.2 we can conclude that f has a unique fixed point in X.

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