Acta Universitatis Apulensis ISSN: 1582-5329 No. 30/2012 pp. 305-314

SOME CONVEXITY PROPERTIES FOR TWO INTEGRAL OPERATORS

Laura Stanciu

Abstract. In this paper we introduce two integral operators for analytic functions $f_i(z)$, $g_i(z)$ in the open unit disk $\mathbb{U}$. The main object of the present paper is to study the order of convexity for these integral operators.

2000 *Mathematics Subject Classification*: Primary 30C45; Secondary 30C75.

Keywords and phrases. Analytic functions; Integral Operators; Starlike functions; Convex functions; General Schwarz Lemma.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.

A function f belonging to $\mathcal{S}$ is a starlike function by the order α, $0 \leq \alpha < 1$ if f satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

We denote this class by $\mathcal{S}^*(\alpha)$.

A function f belonging to $\mathcal{S}$ is a convex function by the order $\alpha, 0 \leq \alpha < 1$ if f satisfies the inequality

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \alpha \qquad (z \in \mathbb{U}).$$

We denote this class by $\mathcal{K}(\alpha)$. A function $f \in \mathcal{S}$ is in the class $\mathcal{R}(\alpha)$ if and only if

$$\operatorname{Re}\left(f'(z)\right) > \alpha \qquad (z \in \mathbb{U}).$$

In [1], Frasin and Jahangiri introduced the class $\mathcal{B}(\mu, \alpha)$ defined as follows.

A function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu, \alpha)$ if and only if

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1 - \alpha \tag{1}$$

$(z \in \mathbb{U}; 0 \leq \alpha < 1; \mu \geq 0)$.

Note that the condition (1) is equivalent to

$$\operatorname{Re}\left(f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right) > \alpha$$

for $(z \in \mathbb{U}; 0 \leq \alpha < 1; \mu \geq 0)$. Clearly, $\mathcal{B}(1,\alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{B}(0,\alpha) = \mathcal{R}(\alpha)$ and $\mathcal{B}(2,\alpha) = \mathcal{B}(\alpha)$ the class which has been introduced and studied by Frasin and Darus [2] (see also [3]). Here, in our present investigation, we introduce two general families of integral operators:

$$I_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\frac{\alpha_i - 1}{M_i}} \left(g_i'(t)\right)^{\gamma_i} dt \tag{2}$$

$\alpha_i, \gamma_i \in \mathbb{C}; f_i, g_i \in \mathcal{A}$, $M_i \geq 1$ for all $i \in \{1, 2, ..., n\}$.

$$J_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\delta_i} \left(e^{g_i(t)}\right)^{\gamma_i} dt \tag{3}$$

$\delta_i, \gamma_i \in \mathbb{C}; f_i, g_i \in \mathcal{A}$ for all $i \in \{1, 2, ..., n\}$.

Remark 1.1. The operator $I_n(z)$ was derived by an operator introduced by Pescar in [5] and studied by Ularu in [6].
In order to prove our main results, we recall the following lemma.

Lemma 1.1.(*General Schwarz Lemma)* [4] *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger or equal to m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \qquad (z \in \mathbb{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2.Main Results

Theorem 2.1. *Let the functions f_i, $g_i \in \mathcal{A}$ for all $i \in \{1, 2, ..., n\}$. For any given $M_i \geq 1$, $N_i \geq 1$ satisfying the conditions*

$$|f_i(z)| \leq M_i \qquad (z \in \mathbb{U}), \qquad \left| \frac{z^2 f_i'(z)}{f_i^2(z)} - 1 \right| \leq \frac{2M_i - 1}{M_i} \qquad (z \in \mathbb{U}) \tag{4}$$

and

$$\left| \frac{z g_i''(z)}{g_i'(z)} - 1 \right| \leq N_i \qquad (z \in \mathbb{U}) \tag{5}$$

there exist numbers $\alpha_i, \gamma_i \in \mathbb{C}$ such that

$$\lambda = 1 - \sum_{i=1}^{n} [3 |\alpha_i - 1| + |\gamma_i| (N_i + 1)]$$

and

$$\sum_{i=1}^{n} [3 |\alpha_i - 1| + |\gamma_i| (N_i + 1)] < 1$$

for all $i \in \{1, 2, ..., n\}$. In these conditions, the integral operator $I_n(z)$ defined by (2) is in $\mathcal{K}(\lambda)$.

Proof. From (2) we obtain

$$I_n'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z} \right)^{\frac{\alpha_i - 1}{M_i}} \left(g_i'(z) \right)^{\gamma_i}$$

and

$$I_n''(z)=\sum_{i=1}^{n}\left[\frac{\alpha_i-1}{M_i}\left(\frac{f_i(z)}{z}\right)^{\frac{\alpha_i-1}{M_i}-1}\left(\frac{zf_i'(z)-f_i(z)}{z^2}\right)\left(g_i'(z)\right)^{\gamma_i}\right]\prod_{\substack{k=1\\k\neq i}}^{n}\left(\frac{f_k(z)}{z}\right)^{\frac{\alpha_k-1}{M_k}}\left(g_k'(z)\right)^{\gamma_k}$$

$$+\sum_{i=1}^{n}\left[\left(\frac{f_i(z)}{z}\right)^{\frac{\alpha_i-1}{M_i}}\gamma_i\left(g_i'(z)\right)^{\gamma_i-1}g_i''(z)\right]\prod_{\substack{k=1\\k\neq i}}^{n}\left(\frac{f_k(z)}{z}\right)^{\frac{\alpha_k-1}{M_k}}\left(g_k'(z)\right)^{\gamma_k}.$$

After the calculus we obtain that

$$\frac{zI_n''(z)}{I_n'(z)}=\sum_{i=1}^{n}\left[\frac{\alpha_i-1}{M_i}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\gamma_i\frac{zg_i''(z)}{g_i'(z)}\right]. \tag{6}$$

It follows from (6) that

$$\left|\frac{zI_n''(z)}{I_n'(z)}\right|\leq\sum_{i=1}^{n}\left[\frac{|\alpha_i-1|}{M_i}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\gamma_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|\right]$$

$$\leq\sum_{i=1}^{n}\left[\frac{|\alpha_i-1|}{M_i}\left(\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|+1\right)\left|\frac{f_i(z)}{z}\right|+1\right)+|\gamma_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|\right]. \tag{7}$$

From the hypothesis (4) of Theorem 2.1., we have

$$|f_i(z)|\leq M_i \qquad (z\in\mathbb{U}) \qquad \text{and} \qquad \left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|\leq\frac{2M_i-1}{M_i} \qquad (z\in\mathbb{U})$$

for all $i\in\{1,2,...,n\}$. By applying the General Schwarz Lemma, we thus obtain

$$|f_i(z)|\leq M_i\,|z| \qquad (z\in\mathbb{U};i\in\{1,2,...,n\}).$$

Using the condition (5) and from the inequality (7), we obtain

$$\left|\frac{zI_n''(z)}{I_n'(z)}\right|\leq\sum_{i=1}^{n}\left[3\,|\alpha_i-1|+|\gamma_i|\left(\left|\frac{zg_i''(z)}{g_i'(z)}-1\right|+1\right)\right]$$

$$\leq\sum_{i=1}^{n}\left(3\,|\alpha_i-1|+|\gamma_i|\,(N_i+1)\right)$$

$$=1-\lambda$$

which implies that $I_n(z) \in \mathcal{K}(\lambda)$.

Setting $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$ in Theorem 2.1., we have

Corollary 2.2. *Let the functions $g_i \in \mathcal{A}$ for all $i \in \{1, 2, ..., n\}$. For any given $N_i \geq 1$ satisfying the condition*

$$\left| \frac{z g_i''(z)}{g_i'(z)} - 1 \right| \leq N_i \qquad (z \in \mathbb{U})$$

there exist $\gamma_i \in \mathbb{C}$ such that

$$\lambda = 1 - \sum_{i=1}^{n} |\gamma_i| (N_i + 1)$$

and

$$\sum_{i=1}^{n} |\gamma_i| (N_i + 1) < 1$$

for all $i \in \{1, 2, ..., n\}$. In these conditions, the integral operator

$$I(g_1, ..., g_n)(z) = \int_0^z \prod_{i=1}^{n} \left(g_i'(t) \right)^{\gamma_i} dt$$

is in $\mathcal{K}(\lambda)$.

Setting $n = 1$ in Theorem 2.1., we have

Corollary 2.3. *Let the functions f, $g \in \mathcal{A}$. For any given $M \geq 1$ and $N \geq 1$ satisfying the conditions*

$$|f(z)| \leq M \qquad (z \in \mathbb{U}), \qquad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq \frac{2M - 1}{M} \qquad (z \in \mathbb{U})$$

and

$$\left| \frac{z g''(z)}{g'(z)} - 1 \right| \leq N \qquad (z \in \mathbb{U})$$

there exist numbers $\alpha, \gamma \in \mathbb{C}$ such that

$$\lambda = 1 - [3 |\alpha - 1| + |\gamma| (N + 1)]$$

and

$$[3 |\alpha - 1| + |\gamma| (N + 1)] < 1.$$

In these conditions, the integral operator

$$I(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\frac{\alpha-1}{M}}\left(g'(t)\right)^{\gamma}dt$$

is in $\mathcal{K}(\lambda)$.

Setting $M=1$ and $N=1$ in Corollary 2.3., we have

Corollary 2.4. *Let the functions* $f,\ g\in\mathcal{A}$. *If*

$$|f(z)|\leq 1\qquad(z\in\mathbb{U}),\qquad\left|\frac{z^2f'(z)}{f^2(z)}-1\right|\leq 1\qquad(z\in\mathbb{U})$$

and

$$\left|\frac{zg''(z)}{g'(z)}-1\right|\leq 1\qquad(z\in\mathbb{U})$$

there exist numbers $\alpha,\gamma\in\mathbb{C}$ *such that*

$$\lambda=1-\left[3\left|\alpha-1\right|+2\left|\gamma\right|\right]$$

and

$$\left[3\left|\alpha-1\right|+2\left|\gamma\right|\right]<1.$$

Then the integral operator

$$I(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha-1}\left(g'(t)\right)^{\gamma}dt$$

is in $\mathcal{K}(\lambda)$.

Theorem 2.5. *Let the functions* $f_i,g_i\in\mathcal{A}$, *where* g_i *be in the class* $\mathcal{B}(\mu_i,\alpha_i)$, $\mu_i\geq 0, 0\leq\alpha_i<1$ *for all* $i\in\{1,2,...,n\}$. *For any given* $\mu_i\geq 0, 0\leq\alpha_i<1$, $M_i\geq 1$ *and* $N_i\geq 1$ *satisfying the conditions*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right|\leq M_i\qquad(z\in\mathbb{U})\qquad\textit{and}\qquad|g_i(z)|\leq N_i\qquad(z\in\mathbb{U})$$

there exist numbers $\delta_i,\gamma_i\in\mathbb{C}$ *such that*

$$\lambda=1-\sum_{i=1}^{n}\left[\left|\delta_i\right|(M_i+1)+\left|\gamma_i\right|(2-\alpha_i)\,{N_i}^{\mu_i}\right]$$

and

$$\sum_{i=1}^{n}[|\delta_i|\,(M_i+1)+|\gamma_i|\,(2-\alpha_i)\,{N_i}^{\mu_i}]<1$$

for all $i\in\{1,2,...,n\}$. In these conditions, the integral operator $J_n(z)$ defined by (3) is in $\mathcal{K}\,(\lambda)$.

Proof. If we make the similar operations to the proof of Theorem 2.1., we have

$$\frac{zJ_n''(z)}{J_n'(z)}=\sum_{i=1}^{n}\left[\delta_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\gamma_i z g_i'(z)\right]. \tag{8}$$

From the relation (8), we obtain that

$$\left|\frac{zJ_n''(z)}{J_n'(z)}\right|\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\gamma_i|\,|zg_i'(z)|\right]$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|\left|\frac{g_i(z)}{z}\right|^{\mu_i}|z|\right]. \tag{9}$$

Since

$$\left|\frac{zf_i'(z)}{f_i(z)}\right|\leq M_i \qquad (z\in\mathbb{U}) \qquad \text{and} \qquad |g_i(z)|\leq N_i \qquad (z\in\mathbb{U})$$

and applying the General Schwarz Lemma for the functions g_i $(i\in\{1,2,...,n\})$, we obtain

$$|g_i(z)|\leq N_i\,|z| \qquad (z\in\mathbb{U})\,. \tag{10}$$

Because $g_i\in\mathcal{B}(\mu_i,\alpha_i)$, $\mu_i\geq 0$, $0\leq\alpha_i<1$ we apply in the relation (9) the inequality (10) and we obtain

$$\left|\frac{zJ_n''(z)}{J_n'(z))}\right|\leq\sum_{i=1}^{n}\left[|\delta_i|\,(M_i+1)+|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|{N_i}^{\mu_i}\right]. \tag{11}$$

From (11) and (1) we obtain

$$\left|\frac{zJ_n''(z)}{J_n'(z)}\right|\leq\sum_{i=1}^{n}\left[|\delta_i|\,(M_i+1)+|\gamma_i|\left(\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}-1\right|+1\right){N_i}^{\mu_i}\right]$$

$$\leq\sum_{i=1}^{n}\left(|\delta_i|\,(M_i+1)+|\gamma_i|\,(2-\alpha_i)\,{N_i}^{\mu_i}\right)$$

$= 1 - \lambda$

which implies that $J_n(z) \in \mathcal{K}(\lambda)$.

Setting $\mu_1 = \mu_2 = ... = \mu_n = 0$, $M_1 = M_2 = ... = M_n = M = 1$ and $N_1 = N_2 = ... = N_n = N = 1$ in Theorem 2.5., we obtain

Corollary 2.6. *Let the functions $f_i, g_i \in \mathcal{A}$, where g_i be in the class $\mathcal{R}(\alpha_i)$, $0 \leq \alpha_i < 1$ for all $i \in \{1, 2, ..., n\}$. For any given $0 \leq \alpha_i < 1$ satisfying the conditions*

$$\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq 1 \quad (z \in \mathbb{U}) \quad \textit{and} \quad |g_i(z)| \leq 1 \quad (z \in \mathbb{U})$$

there exist numbers $\delta_i, \gamma_i \in \mathbb{C}$ such that

$$\lambda = 1 - \sum_{i=1}^{n} [2 |\delta_i| + |\gamma_i| (2 - \alpha_i)]$$

and

$$\sum_{i=1}^{n} [2 |\delta_i| + |\gamma_i| (2 - \alpha_i)] < 1$$

for all $i \in \{1, 2, ..., n\}$. In these conditions, the integral operator $J_n(z)$ defined by (3) is in $\mathcal{K}(\lambda)$.

Setting $\mu_1 = \mu_2 = ... = \mu_n = 1$, $M_1 = M_2 = ... = M_n = M$ and $N_1 = N_2 = ... = N_n = N$ in Theorem 2.5., we obtain

Corollary 2.7. *Let the functions $f_i, g_i \in \mathcal{A}$, where g_i be in the class $\mathcal{S}^*(\alpha_i)$, $0 \leq \alpha_i < 1$ for all $i \in \{1, 2, ..., n\}$. For any given $0 \leq \alpha_i < 1$, $M \geq 1$ and $N \geq 1$ satisfying the conditions*

$$\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq M \quad (z \in \mathbb{U}) \quad \textit{and} \quad |g_i(z)| \leq N \quad (z \in \mathbb{U})$$

there exist $\delta_i, \gamma_i \in \mathbb{C}$ such that

$$\lambda = 1 - \sum_{i=1}^{n} (|\delta_i| (M + 1) + |\gamma_i| (2 - \alpha_i) N)$$

and

$$\sum_{i=1}^{n} (|\delta_i| (M + 1) + |\gamma_i| (2 - \alpha_i) N) < 1$$

for all $i \in \{1, 2, ..., n\}$. *Then the integral operator* $J_n(z)$ *defined by (3) is in* $\mathcal{K}(\lambda)$.

Setting $n = 1$ in Theorem 2.5., we obtain

Corollary 2.8. *Let the functions* $f, g \in \mathcal{A}$, *where* g *be in the class* $\mathcal{B}(\mu, \alpha)$, $\mu \geq 0, 0 \leq \alpha < 1$. *For any given* $\mu \geq 0, 0 \leq \alpha < 1$, $M \geq 1$ *and* $N \geq 1$ *satisfying the conditions*

$$\left| \frac{zf'(z)}{f(z)} \right| \leq M \qquad (z \in \mathbb{U}) \qquad \textit{and} \qquad |g(z)| \leq N \qquad (z \in \mathbb{U})$$

there exist numbers $\delta, \gamma \in \mathbb{C}$ *such that*

$$\lambda = 1 - [|\delta| (M + 1) + |\gamma| (2 - \alpha) N^{\mu}]$$

and

$$[|\delta| (M + 1) + |\gamma| (2 - \alpha) N^{\mu}] < 1.$$

In these conditions, the integral operator

$$J(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\delta} \left(e^{g(t)} \right)^{\gamma} dt$$

is in $\mathcal{K}(\lambda)$.

Acknowledgement. This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

References

[1] B. A. Frasin, J. Jahangiri, *A new and comprehensive class of analytic functions*, Anal. Univ. Ordea Fasc. Math., XV(2008), 59-62.

[2] B. A. Frasin and M.Darus, *On certain analytic univalent functions*, Internat. J. Math. and Math. Sci.25 (5) (2001), 305-310.

[3] B. A. Frasin, *A note on certain analytic and univalent functions*, Southeast Asian J. Math. 28 (2004), 829-836.

[4] Z. Nehari, *Conformal mapping*, McGraw-Hill Book Comp., New York, 1952.

[5] V. Pescar, *The univalence of an integral operator*, General Mathematics, Vol. 19, No 4(2011), 69-74.

[6] N. Ularu, *Convexity properties for an integral operator*, Acta Universitatis Apulensis, no 27(2011), pp. 115-120.

Laura Filofteia Stanciu
University of Piteşti
Department of Mathematics
Târgul din Vale Str., No.1, 110040, Piteşti, Argeş,
România;
email: *laura_stanciu_30@yahoo.com*