THERMAL CONDUCTION IN GRIDWORKS
(CYLINDRICAL DOMAINS)

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Abstract. In this paper we study a stationary thermal problem on gridworks, characterized by two small parameters: $\varepsilon$- period and $\delta$- thickness distributed along the structure layers.

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1. Introducere

We now consider a particular case of three-dimensional lattice structures, the gridworks that consist in regular array of thin wires.

The specifics of these structures consists in just following two-way periodicity $ox_1$ and $ox_2$. The two small parameters on which the structure are $\varepsilon$- during which distribute the reference cell and $\delta$- small thickness of the material distributed along the structure cross section. There is also a third parameter $e$, which is the thickness of the structure in longitudinal section. In our case $e$ and $\varepsilon$ have the same order $e = k\varepsilon$.

The novelty of our problem consists in reticulated structure plate geometry: the period that we’re on the covers is made up of horizontal bars, vertical and oblique. Due to this the limit problem obtained after the homogenization by two parameters $\varepsilon$ and $\delta$ is new and at the same time simple: we started from a thermal problem on a heterogeneous domain which depends on $\varepsilon$ and $\delta$, and we have a two-dimensional problem with partial differential second order with constant and elliptical coefficients.

Homogenization reduces initial problem to two simple problems: one on the cell of periodicity and another one on a fixed domain without holes.

In the first stage we use a result obtained in [1]. Here, applying the variational method of Tartar, homogenized the initial equation after $\varepsilon \to 0$.

In the second stage we got our new result, using the method of dilation introduced in [2]. Dilatation technique we use is to changes the appropriate variables that transforms bars $H_\delta, V_\delta, O_\delta^1$ and $O_\delta^2$ in the entire cell reference $Y$. 

185
In our result, homogenized coefficients obtained by $\delta \to 0$ are simple algebraic combinations between the characteristics of the material. The problem obtained in theorem 2, can be solved explicit. We should note the following aspect: the crossing of the boundary after $\varepsilon \to 0$, then $\delta \to 0$ we get homogenized coefficients that depends strictly reticulated structure, namely the periodicity cell of structure.

2. THE GEOMETRY OF THE STRUCTURE

Let $\omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ and $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$, and $\omega$ is covered periodically with the reference cell $Y = (0, 1) \times (0, 1)$. We have $\frac{L_1}{\varepsilon}, \frac{L_2}{\varepsilon} \in \mathbb{Q}$ and we choose $\varepsilon$ such that $N^1_\varepsilon = \frac{L_1}{\varepsilon}, N^2_\varepsilon = \frac{L_2}{\varepsilon}$ to be integer numbers. $\varepsilon$ is called the period which is distributed $Y$ in $\omega$. In Figure 1 represent the periodicity cell $Y_\delta$ defined by:

$$Y_\delta = \left\{ y \in Y \left| \text{dist} (y, \partial Y) < \frac{\delta}{2} \right. \right\} \cup O^1_\delta \cup O^2_\delta$$

where $O^1_\delta, O^2_\delta$ are rectangular slash of length $\sqrt{2}$ and thickness $\delta$.

We define the hole $T_\delta = Y \setminus Y_\delta$.

Figure 1: The cell $Y_\delta$.

$Y_\delta$ is occupied by the material from the cell $Y$. We consider $\omega_{\varepsilon \delta}$ the perforated area from $\omega$ or occupied by the material from $\omega$ after distribution of the periodicity cell $Y_\delta$ with period $\varepsilon$ by two directions $\alpha x_1$ and $\alpha x_2$. The domain $\omega_{\varepsilon \delta}$ has $N^1_\varepsilon \times N^2_\varepsilon$ holes which do not intersects the border of $\omega$.

Consider three-dimensional perforated domain $\Omega^\varepsilon_{\varepsilon \delta} = \omega_{\varepsilon \delta} \times (-\varepsilon, \varepsilon)$ which is a gridworks type plates which depends on three small parameters: $\varepsilon$ the period, $e$ the plate thickness and $\delta$ thickness of the bars (oblique, horizontal and vertical) which forms covers. The periodicity cell of the structure is $Y_\delta = Y_\delta \times \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Because the correctors $w^{\delta k}_\alpha, w^{\delta k}_3$ that appear in Theorem 1 are $Y$-periodic, choose - for ease of calculations that appear in the method of dilation of theorem 2 of this article - the next cell reference $Y = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and the periodicity cell $Y_\delta$ shown in Figure 2, is defined by
\[ Y_\delta = H_\delta \cup V_\delta \cup O_1^\delta \cup O_2^\delta, \]

where the bars \( O_1^\delta \) and \( O_2^\delta \) are the same as Figure 1, and
\[
H_\delta = \{ y \in Y \mid |y_1| \leq \frac{1}{2}, |y_2| \leq \frac{\delta}{2} \},
\]
\[
V_\delta = \{ y \in Y \mid |y_1| \leq \frac{\delta}{2}, |y_2| \leq \frac{1}{2} \}.
\]

![Figure 2: The period \( Y_\delta \).](image)

3. Statement of the problem

Let the stationary temperature problem on \( \Omega^{e\delta} \):

\[
\begin{aligned}
- \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u^{e\delta}}{\partial x_j} \right) &= f^e \text{ in } \Omega^{e\delta} \\
 a_{3j} \frac{\partial u^{e\delta}}{\partial x_j} n_3 &= g^e \pm \text{ in } \Gamma^{e\pm} \\
 a_{\alpha j} \frac{\partial u^{e\delta}}{\partial x_j} n_\alpha &= 0 \text{ on } \delta \Gamma^{e\delta} \\
u^{e\delta} &= 0 \text{ on } \Gamma_0
\end{aligned}
\]

where:

\[
(1)
\]
The assumptions:

\( \Gamma^{\pm}_{e,d} = \omega_{e,d} \times \{ \pm \frac{\varepsilon}{2} \} \), are the two covers on the structure

\( T_{\varepsilon,d} = \omega \setminus \omega_{e,d}, \) the set of holes

\( T_{\varepsilon,d}^{e} = T_{\varepsilon,d} \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), \)

\( \Gamma^{e}_{0} = \delta \omega \times (-\frac{1}{2}, \frac{1}{2}), \) external border of the structure

and we make the assumptions:

1. \( f_{e}^{\varepsilon} \in C^{1}(R^{3}) \cap L^{2}(\Omega) \) and \( g_{e}^{\varepsilon} \in C^{1}(R^{2}) \cap L^{2}(\omega) \).

2. There is a constant \( A > 0 \) such that: \( a_{ij}\xi_{i}\xi_{j} \geq A\xi_{i}\xi_{j}, \forall \xi \in R^{3} \).

Consider the case \( e = k\varepsilon \), so when the period and the plate thickness are the same power.

We are making changes of variables and functions:

\( z_{1} = x_{1}, \ z_{2} = x_{2}, \ z_{3} = \frac{x_{3}}{k}, \)

\( u^{e,\varepsilon}(x_{1}, x_{2}, x_{3}) = u^{e,\varepsilon}(z_{1}, z_{2}, k\varepsilon z_{3}) = u^{e,\varepsilon}(z_{1}, z_{2}, z_{3}); \)

\( f^{e}_{k}(z_{1}, z_{2}, k\varepsilon z_{3}) = f_{k}(z_{1}, z_{2}, z_{3}); \)

\( g^{e,\varepsilon}(z_{1}, z_{2}, k\varepsilon z_{3}) = g^{e,\varepsilon}(z_{1}, z_{2}, z_{3}). \)

\( \Omega^{e}_{\varepsilon,d} \) passes into \( \Omega^{\varepsilon}_{\varepsilon,d} = \omega_{\varepsilon,d} \times (-\frac{1}{2}, \frac{1}{2}); \)

\( \Gamma^{e,\varepsilon}_{\varepsilon,d} \) passes into \( \Gamma^{\varepsilon}_{0} = \delta \omega \times (-\frac{1}{2}, \frac{1}{2}), \) and \( \Omega^{\varepsilon} \) in \( \Omega = \omega \times (-\frac{1}{2}, \frac{1}{2}). \)

After the change the variable and function, problem (1) is written variational:

\[
\int_{\Omega_{\varepsilon,d}} \left[ a_{ij} \frac{\partial u^{e,\varepsilon}_{i}}{\partial z_{i}} \frac{\partial v}{\partial z_{i}} + (k\varepsilon) - 1 \left( a_{ij} \frac{\partial u^{e,\varepsilon}_{ij}}{\partial z_{i}} \frac{\partial v}{\partial z_{i}} + a_{3i} \frac{\partial u^{e,\varepsilon}_{i}}{\partial z_{i}} \frac{\partial v}{\partial z_{i}} \right) \right] dz = \int_{\Gamma^{e,\varepsilon}_{\varepsilon,d}} f^{e}_{k} v dz + (k\varepsilon)^{-1} \int_{\Gamma^{e,\varepsilon}_{\varepsilon,d}} g^{e,\varepsilon}_{k} v dz_{1} dz_{2} + (k\varepsilon)^{-1} \int_{\Gamma^{e,\varepsilon}_{\varepsilon,d}} g^{e,\varepsilon}_{k} v dz_{1} dz_{2},
\]

for all

\( v \in V_{\varepsilon,d} = \{ v \in H^{1}(\Omega_{\varepsilon,d}) : v = 0 \text{ on } \partial \omega \times (-\frac{1}{2}, \frac{1}{2}) \} \)

and

\( \| v \|_{V_{\varepsilon,d}} = \left[ \sum_{i=1}^{3} \int_{\Omega_{\varepsilon,d}} \left( \frac{\partial v}{\partial z_{i}} \right)^{2} dx \right]^{1/2}. \)

4. The homogenization of the problem

First we do \( \varepsilon \to 0 \) and \( \delta \) consider fixed.

After applying Tartar’s variational method [3], we find:

**Theorem 1.** Consider the following assumptions:

\[
(f_{k}^{e}_{k} \chi_{\Omega_{\varepsilon,d}})^{\varepsilon \to 0} \text{ weak in } L^{2}(\Omega),
\]

\[(k\varepsilon)^{-1} (g^{e,\varepsilon}_{k} \chi_{\omega_{\varepsilon,d}})^{\varepsilon \to 0} \text{ weak in } L^{2}(\omega).\]

Then there is an extension operator \( P^{e,\varepsilon} \in L(V_{\varepsilon,d}; H^{1}_{0}(\Omega)) \) such that
\(P^{\delta}u_k^{\delta} \to [\varepsilon \to 0]u_k^{\delta} \) weak in \(H^1(\Omega)\) where \(u_k^{\delta} = u_k^{\delta}(z_1, z_2)\), \(u_k^{\delta} \in H^1_0(\omega)\) satisfies the problem

\[
\begin{cases}
-q_{\alpha\beta}^{\delta} \frac{\partial^2 u_k^{\delta}}{\partial z_\alpha \partial z_\beta} = \frac{\text{meas } Y}{\text{meas } Y} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} f_k^{*} (z_1, z_2, z_3) \, dz_3 + \frac{\text{meas } Y}{\text{meas } Y} \left( g_k^{*+} + g_k^{*-} \right) & \text{in } \omega \\
\frac{\partial u_k^{\delta}}{\partial n} = 0 & \text{on } \partial \omega
\end{cases}
\]  

(4)

where the homogenized coefficients are:

\[
q_{\alpha\beta}^{\delta} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Y} \left( a_\gamma a_\beta \frac{\partial u_\alpha^{\delta k}}{\partial y_\gamma} + k^{-1} a_3 a_3 \frac{\partial u_\alpha^{\delta k}}{\partial y_3} \right) dy
\]  

(5)

where the correction functions \(u_\alpha^{\delta k}\) satisfies the problem:

\[
\begin{cases}
-k^{-2} \frac{\partial}{\partial y_3} \left( a_3 a_3 \frac{\partial u_\alpha^{\delta k}}{\partial y_3} \right) = 0 & \text{in } Y \times \left( -\frac{1}{2}, \frac{1}{2} \right) \\
\left( a_\gamma a_\beta \frac{\partial u_\alpha^{\delta k}}{\partial y_\gamma} + k^{-1} a_3 a_3 \frac{\partial u_\alpha^{\delta k}}{\partial y_3} \right) n_j = 0 & \text{on } \left( \partial T_3 \times \left( -\frac{1}{2}, \frac{1}{2} \right) \right) \cup \left( \partial Y \times \left\{ \pm \frac{1}{2} \right\} \right)
\end{cases}
\]  

(6)

\(w_\alpha^{\delta k} - y_\alpha\) is periodic in \(y_1\) and \(y_2\).

Further, we do \(\delta \to 0\), and using the dilatation method, find:

**Theorem 2.** We have \(u_k^{\delta} \to [\delta \to 0]u_k^{*} \) weak in \(H^1_0(\omega)\), where:

\[
\begin{cases}
-q_{\alpha\beta}^{*} \frac{\partial^2 u_k^{*}}{\partial z_\alpha \partial z_\beta} = 2 \left( 1 + \sqrt{2} \right) \left[ \int_{-\frac{1}{2}}^{\frac{1}{2}} f_k^{*} (z_1, z_2, z_3) \, dz_3 + \left( g_k^{*+} + g_k^{*-} \right) \right] & \text{in } \omega \\
u_k^{*} = 0 & \text{on } \partial \omega
\end{cases}
\]  

(7)

where the coefficients \(q_{\alpha\beta}^{*}\) are elliptical and are given by:

\[
\begin{align*}
q_{11}^{*} &= D \left[ \frac{1}{A_{22}} + \frac{\sqrt{2}}{a_{11} - a_{13} - a_{31} + a_{33}} + \frac{\sqrt{2}}{a_{11} + a_{22} + a_{13} + a_{31} + a_{22} + a_{33}} \right] \\
q_{22}^{*} &= D \left[ \frac{1}{A_{11}} + \frac{\sqrt{2}}{a_{11} - a_{13} - a_{31} + a_{33}} + \frac{\sqrt{2}}{a_{11} + a_{22} + a_{13} + a_{31} + a_{22} + a_{33}} \right] \\
q_{12}^{*} = q_{21}^{*} &= \sqrt{2} D \left[ \frac{1}{a_{11} - a_{13} - a_{31} + a_{33}} - \frac{1}{a_{11} + a_{22} + a_{33} + a_{31}} \right]
\end{align*}
\]  

(8)

where:

\(D = \text{det } A\), and \(A_{11}, A_{22}\) are algebraic complements.
References


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