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NEW TYPE OF CHEBYCHEV-GRSS INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper we will show some new inequalities for convex functions, and we will also make a connection between it and Grüss inequality, which implies the existence of new class of functions satisfied Grüss inequality.

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Introduction and Main Results

In 1935, G. Grüss (see [2]) proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)\right| \leqslant \frac{1}{4}(\Gamma-\gamma)(\Psi-\phi) \quad (1.1)$$

where $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\gamma\leqslant f(x)\leqslant\Gamma,\ \ \phi\leqslant g(x)\leqslant\Psi$$

for each $x\in[a,b]$, where γ,ϕ,Γ,Ψ are given real constants. Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one. For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see [5, Chapter X] and the papers [2,6].
The inequality (1.1) has evoked the interest of many researchers and numerous generalizations, variants and extensions have appeared in the literature, to mention a few, see [1,3,7,8] and the references cited therein.

In [4] the second author and B. Belaïdi proved new type of Chebychev's inequality for convex functions and they obtained the following results:

Theorem A *Let* $f, g : [a,b] \longrightarrow \mathbb{R}$ *be convex* (*or concave*) *functions and* $p : [a,b] \to \mathbb{R}_+$ *be integrable symmetric function about* $x = \frac{a+b}{2}$ (i. e. $p(a+b-x) = p(x)$, *for all* $x \in [a,b]$), *then*

$$\int_a^b p(x) f(x) g(x) dx + \int_a^b p(x) f(x) g(a+b-x) dx$$
$$\geqslant \frac{2}{\int_a^b p(x) dx} \int_a^b p(x) f(x) dx \int_a^b p(x) g(x) dx. \tag{1.2}$$

If f *is convex* (*or concave*) *and* g *is concave* (*or convex*) *functions, then the inequality* (1.2) *is reversed, equality in* (1.2) *holds if and only if either* g *or* f *is constant almost everywhere.*

Theorem B *Let* $f, g : [a,b] \longrightarrow \mathbb{R}$ *be convex* (*or concave*) *functions. If* g *is symmetric function about* $x = \frac{a+b}{2}$, *then*

$$\int_a^b f(x) g(x) dx \geqslant \frac{1}{b-a} \int_a^b f(x) dx \int_a^b g(x) dx. \tag{1.3}$$

If f *is convex* (*or concave*) *and* g *is concave* (*or convex*) *functions, then the inequality* (1.3) *is reversed, equality in* (1.3) *holds if and only if either* g *or* f *is constant almost everywhere.*

Theorem C *Let* $f, g : [a,b] \longrightarrow \mathbb{R}$ *where* f *is convex function and* g *decreasing in* $\left[a, \frac{a+b}{2}\right]$ *and increasing in* $\left[\frac{a+b}{2}, b\right]$, *then the inequality* (1.3) *holds.*

The aim of this paper is to proved a new version of Grüss inequality for convex functions and find a new class of functions satisfies Grüss inequality, before we stat our results we denote by

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx + \frac{1}{b-a}\int_a^b f(x)g(a+b-x)\,dx - \frac{2}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx, \tag{1.4}$$

and

$$\rho_f = \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)$$

and we obtain the following results:

Theorem 1.1 *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be convex $($or concave$)$ functions, then*

$$0 \leqslant T(f,g) \leqslant \frac{1}{2}\rho_f\rho_g, \tag{1.5}$$

where the constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller one. If f is convex $($or concave$)$ and g is concave $($or convex$)$ functions, then the inequality (1.5) *is reversed.*

Corollary 1.1 *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be convex $($or concave$)$ functions such that*

$$\gamma \leqslant f(x) \leqslant \Gamma,\ \ \phi \leqslant g(x) \leqslant \Psi$$

for each $x\in[a,b]$, where γ,ϕ,Γ,Ψ are given real constants. Then

$$T(f,g) \leqslant \frac{1}{2}\rho_f\rho_g \leqslant \frac{1}{2}(\Gamma-\gamma)(\Psi-\phi). \tag{1.6}$$

where the constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller one.

Corollary 1.2 *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be convex $($or concave$)$ functions. If either f or g is symmetric function about $x=\frac{a+b}{2}$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$
$$\leqslant \frac{1}{4}\rho_f\rho_g. \tag{1.7}$$

Furthermore, if

$$\gamma\leqslant f(x)\leqslant\Gamma,\ \ \phi\leqslant g(x)\leqslant\Psi \tag{1.8}$$

for each $x\in[a,b]$, *where* γ,ϕ,Γ,Ψ *are given real constants, then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$
$$\leqslant \frac{1}{4}\rho_f\rho_g$$
$$\leqslant \frac{1}{4}(\Gamma-\gamma)(\Psi-\phi). \tag{1.9}$$

where the constant $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller one.*

Corollary 1.3 *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *be convex (or concave) function and symmetric about* $x=\frac{a+b}{2}$, *then*

$$\frac{1}{b-a}\int_a^b f^2(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2\leqslant\frac{1}{4}\rho_f^2. \tag{1.10}$$

Theorem 1.2 *Let* $f,g:[a,b]\longrightarrow\mathbb{R}$ *where* f *is convex function and* g *decreasing on* $\left[a,\frac{a+b}{2}\right]$ *and increasing on* $\left[\frac{a+b}{2},b\right]$, *then the inequality* (1.5) *holds. If* f *is convex function and* g *increasing on* $\left[a,\frac{a+b}{2}\right]$ *and decreasing on* $\left[\frac{a+b}{2},b\right]$, *then the inequality* (1.5) *reversed.*

Corollary 1.4 *Let* $f,g:[a,b]\longrightarrow\mathbb{R}$ *where* f *is convex function and* g *decreasing on* $\left[a,\frac{a+b}{2}\right]$ *and symmetric about* $x=\frac{a+b}{2}$, *then the inequality* (1.7) *holds. If* f *is convex function and* g *increasing in* $\left[a,\frac{a+b}{2}\right]$ *and symmetric about* $x=\frac{a+b}{2}$, *then the inequality* (1.7) *reversed.*

2. Lemmas

Lemma 2.1 *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *be convex (or concave) function, then*

$$F(x)=\frac{1}{2}\left(f(x)+f(a+b-x)\right) \tag{2.1}$$

satisfy the following:
(1) *F is convex (or concave) function.*
(2) *For all $x\in[a,b]:F\left(\frac{a+b}{2}\right)\underset{\geq}{\leqslant}F(x)\underset{\geq}{\leqslant}F(a)=F(b)$.*

Proof: (1) Let f be a convex function. For all $x,y\in[a,b]$ and $\lambda\in[0,1]$ we have

$$\begin{aligned}F(\lambda x+(1-\lambda)y) &= \frac{1}{2}\left(f(\lambda x+(1-\lambda)y)+f(\lambda(a+b-x)+(1-\lambda)(a+b-y))\right)\\ &\leqslant \frac{1}{2}\left(\lambda f(x)+(1-\lambda)f(y)+\lambda f(a+b-x)+(1-\lambda)f(a+b-y)\right)\\ &= \lambda\left(\frac{1}{2}\left(f(x)+f(a+b-x)\right)\right)\\ &\quad+(1-\lambda)\left(\frac{1}{2}\left(f(y)+f(a+b-y)\right)\right)\\ &= \lambda F(x)+(1-\lambda)F(y).\end{aligned}$$

Hence F is convex function.

(2) Let f be a convex function, we have

$$F\left(\frac{a+b}{2}\right)=F\left(\frac{a+b-x+x}{2}\right)\leqslant\frac{1}{2}F(x)+\frac{1}{2}F(a+b-x)=F(x) \tag{2.2}$$

and

$$F(x)=F\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\leqslant\frac{x-a}{b-a}F(b)+\frac{b-x}{b-a}F(a)=F(a). \tag{2.3}$$

Lemma 2.2 [4] *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *be convex (or concave) function, then*

$$F(x)=\frac{1}{2}\left(f(x)+f(a+b-x)\right)$$

is decreasing (increasing) on $\left[a,\frac{a+b}{2}\right]$ and increasing (decreasing) on $\left[\frac{a+b}{2},b\right]$.

Proof: Suppose that f is convex function and using the same proof for concave functions. Let $x, y \in \left[a, \frac{a+b}{2}\right]$ such that $x \leqslant y$, then there exists $\lambda \in [0,1]$ such that $y = \lambda x + (1-\lambda)\frac{a+b}{2}$. Since F is convex function then we have

$$\begin{aligned} F(y) &= F\left(\lambda x + (1-\lambda)\frac{a+b}{2}\right) \\ &\leqslant \lambda F(x) + (1-\lambda) F\left(\frac{a+b}{2}\right) \\ &= F(x) + (1-\lambda)\left(F\left(\frac{a+b}{2}\right) - F(x)\right), \end{aligned}$$

by Lemma 2.1 we get $F(y) \leqslant F(x)$ then F is decreasing in $\left[a, \frac{a+b}{2}\right]$. Now, let $x, y \in \left[\frac{a+b}{2}, b\right]$ such that $x \leqslant y$, then there exist $\lambda \in [0,1]$ such that $x = \lambda y + (1-\lambda)\frac{a+b}{2}$. Since F is convex function we have

$$\begin{aligned} F(x) &= F\left(\lambda y + (1-\lambda)\frac{a+b}{2}\right) \leqslant \lambda F(y) + (1-\lambda) F\left(\frac{a+b}{2}\right) \\ &= F(y) + (1-\lambda)\left(F\left(\frac{a+b}{2}\right) - F(y)\right), \end{aligned}$$

by Lemma 2.1 we get $F(x) \leqslant F(y)$ then F is increasing in $\left[\frac{a+b}{2}, b\right]$

3. Proof of Theorems

Proof of Theorem 1.1: First, without loss of generality we suppose that f and g are convex functions and we denote by F and G the following functions

$$F(x) = \frac{1}{2}\left(f(x) + f(a+b-x)\right),$$

$$G(x) = \frac{1}{2}\left(g(x) + g(a+b-x)\right).$$

Since f and g are convex functions, and by using Lemma 2.2 and Lemma 2.1 we deduce that F and G having the same variation and

$$F\left(\frac{a+b}{2}\right) \leqslant F(x) \leqslant F(a) = F(b),$$

$$G\left(\frac{a+b}{2}\right) \leqslant G(x) \leqslant G(a) = G(b),$$

for each $x \in [a,b]$. Then by applying Grüss inequality for F and G and by using Chebychev's inequality, we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b F(x)\,G(x)\,dx-\frac{1}{(b-a)^2}\int_a^b F(x)\,dx\int_a^b G(x)\,dx\right|\\
=\;&\frac{1}{b-a}\int_a^b F(x)\,G(x)\,dx-\frac{1}{(b-a)^2}\int_a^b F(x)\,dx\int_a^b G(x)\,dx\\
\leqslant\;&\frac{1}{4}\left(F(a)-F\left(\frac{a+b}{2}\right)\right)\left(G(a)-G\left(\frac{a+b}{2}\right)\right) \qquad (3.1)
\end{aligned}$$

which we can write as

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b [f(x)\,g(x)\,dx+f(a+b-x)\,g(a+b-x)]\,dx\\
&+\frac{1}{b-a}\int_a^b [f(x)\,g(a+b-x)\,dx+f(a+b-x)\,g(x)]\,dx\\
&-\frac{1}{(b-a)^2}\left(\int_a^b [f(x)+f(a+b-x)]\,dx\right)\\
&\left(\int_a^b [g(x)+g(a+b-x)]\,dx\right)\\
\leqslant\;&\frac{1}{4}\left(F(a)-F\left(\frac{a+b}{2}\right)\right)\left(G(a)-G\left(\frac{a+b}{2}\right)\right) \qquad (3.2)
\end{aligned}$$

Using the identity

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx, \qquad (3.3)$$

and

$$\int_a^b f(x)\,g(a+b-x)\,dx=\int_a^b f(a+b-x)\,g(x)\,dx. \qquad (3.4)$$

We obtain

$$\begin{aligned} & \frac{2}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{2}{b-a}\int_a^b f(x)\,g(a+b-x)\,dx \\ & -\frac{4}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right) \\ \leqslant\ & \frac{1}{4}\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right)\left(g(a)+g(b)-2g\left(\frac{a+b}{2}\right)\right) \end{aligned}$$

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which is equivalent to

$$T(f,g) \leqslant \frac{1}{2}\rho_f\rho_g. \tag{3.6}$$

Now, suppose that f is convex function and g is concave function, we deduce that F and G are oppositely ordered and by using Lemma 2.1, we have

$$F\left(\frac{a+b}{2}\right) \leqslant F(x) \leqslant F(a) = F(b),$$

$$G(a) = G(b) \leqslant G(x) \leqslant G\left(\frac{a+b}{2}\right),$$

for each $x \in [a,b]$. Then by applying Grüss inequality for F and G and by using Chebychev's inequality, we obtain

$$\begin{aligned} & \left|\frac{1}{b-a}\int_a^b F(x)\,G(x)\,dx - \frac{1}{(b-a)^2}\int_a^b F(x)\,dx\int_a^b G(x)\,dx\right| \\ = & -\left(\frac{1}{b-a}\int_a^b F(x)\,G(x)\,dx - \frac{1}{(b-a)^2}\int_a^b F(x)\,dx\int_a^b G(x)\,dx\right) \\ \leqslant & -\frac{1}{4}\left(F(a)-F\left(\frac{a+b}{2}\right)\right)\left(G(a)-G\left(\frac{a+b}{2}\right)\right). \end{aligned} \tag{3.7}$$

By same reasoning as above

$$T(f,g) \geqslant \frac{1}{2}\rho_f\rho_g \tag{3.8}$$

and the proof of Theorem 1.1 is complete.

Proof of Corollary 1.1: First, without loss of generality we suppose that f and g are convex functions. By Theorem 1.1 we have

$$T(f,g) \leq \frac{1}{2}\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right)\left(\frac{g(a)+g(b)}{2} - g\left(\frac{a+b}{2}\right)\right). \quad (3.9)$$

Since

$$\gamma \leqslant f(x) \leqslant \Gamma,\ \phi \leqslant g(x) \leqslant \Psi,$$

for each $x \in [a,b]$, where $\gamma, \phi, \Gamma, \Psi$ are given real constants, then we have

$$0 \leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \Gamma - \gamma, \quad (3.10)$$

and

$$0 \leq \frac{g(a)+g(b)}{2} - g\left(\frac{a+b}{2}\right) \leq \Psi - \phi, \quad (3.11)$$

which implies that

$$\frac{1}{2}\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right)\left(\frac{g(a)+g(b)}{2} - g\left(\frac{a+b}{2}\right)\right) \leq \frac{1}{2}(\Gamma-\gamma)(\Psi-\phi),$$

and the proof of Corollary 1.1 is complete.

Proof of Corollary 1.2: First, without loss of generality we suppose that f and g are convex functions and f is symmetric about $x = \frac{a+b}{2}$. Then

$$f(x) = f(a+b-x), \quad (3.12)$$

for all $x \in [a,b]$. Applying Theorem 1.1 and Corollary 1.1 we obtain (1.7) and (1.9).

Proof of Corollary 1.3: By setting $f(x) = g(x)$ in Theorem 1.1, we obtain (1.10).

Proof of Theorem 1.2: We denote by F and G the following functions

$$F(x) = f(x) + f(a+b-x),$$
$$G(x) = g(x) + g(a+b-x).$$

Since f is convex functions, then by Lemma 2.1, F is decreasing on $\left[a, \frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2}, b\right]$. In order to prove (1.4) we need to prove that G is decreasing on $\left[a, \frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2}, b\right]$.
Let $x, y \in \left[a, \frac{a+b}{2}\right]$, suppose that $x^* = a+b-x$ and $y^* = a+b-y$ where $x^*, y^* \in \left[\frac{a+b}{2}, b\right]$.

It's clear that if $x \leqslant y$, then $x^* \geqslant y^*$. Since g is decreasing in $\left[a, \frac{a+b}{2}\right]$ and increasing in $\left[\frac{a+b}{2}, b\right]$, then we have

$$g(x) \geqslant g(y), \tag{3.13}$$

and

$$g(x^*) \geqslant g(y^*). \tag{3.14}$$

Then

$$G(x) = g(x) + g(x^*) \geqslant g(y) + g(y^*) = G(y), \tag{3.15}$$

which implies that G is decreasing on $\left[a, \frac{a+b}{2}\right]$, by the same method we can prove easily that G is increasing on $\left[\frac{a+b}{2}, b\right]$.
Then we have F and G having the same variation and

$$F\left(\frac{a+b}{2}\right) \leqslant F(x) \leqslant F(a) = F(b), \tag{3.16}$$

$$G\left(\frac{a+b}{2}\right) \leqslant G(x) \leqslant G(a) = G(b), \tag{3.17}$$

and by applying Theorem 1.1, we obtain inequality (1.4).
For the case when f is convex function and g is increasing in $\left[a, \frac{a+b}{2}\right]$ and decreasing in $\left[\frac{a+b}{2}, b\right]$, we use the same reasoning as above.

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