Proceedings of the *International Conference on Theory and Application of Mathematics and Informatics ICTAMI 2005* - Alba Iulia, Romania

COUNTING SOME MULTIDIMENSIONAL INTERPOLATION SCHEMES

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Abstract. In this article we handle the problem of counting the (Z, S, A) Birkhoff uniform interpolation schemes in certain particular cases. More concrete, we have an answer to the following question: giving and particular, what is the number of inferior sets for which (Z, S, A) becomes regular? The case being considered is when $Z = \{(x_1, y_1), (x_2, y_2)\} \subset R^2$ and has at most three elements.

Among the notions we use in this article we mention:

- inferior sets (def. 1, page 77, [1]),
- Birkhoff interpolation schemes (def. 4, page 102, [2]),
- regular interpolation scheme (def. 11, page 162, [4])
- incidence matrix support(8 for [2], page 103) and
- Pólya condition (8, page 62, [3]).

To these notions we add:

1. If (Z, S, A) is a Birkhoff interpolation scheme and if on each interpolation knot it is interpolated the same set of derivatives (A) then the resulted scheme is written (Z, S, A) and is named uniform Birkhoff interpolation scheme.

2. By definition the number $s(Z, A)$ is the cardinality of set {S:(Z, S, A) is regular} , i.e $s(Z, A) = |$ {S:(Z, S, A) is regular}$|$. Also if r is the number of the elements from A then

$$a_r(Z) = |\{A : s(Z, A) \neq 0\}| \ ,$$

and the formal powers series

$$a(Z) = \sum_{r=0}^{\infty} a_r(Z) t^r \in R[[t]]$$

is the "counting series" of the interpolated derivatives associated to Z in the regular schemes (Z, S, A).

Knowledgement of these numbers (even only in particular cases) would bring a much better understanding of the general phenomena specific to the multidimensional interpolation schemes. But we expect to have a very complicated process of computation of these numbers, reaching some difficulties which cross the limits of interpolation theory. Because of this fact it is intended to compute these numbers when the starting points (Z and r) are small and to try to establish different inferior or superior limits of these sets, or different recurrence relations which they satisfy.

The most useful method for obtaining some estimations (or even precise formula) of these numbers is to use the Polya condition which obviously restricts the number of possibilities for choosing the set A when Z is given.

In this article we intend to estimate the numbers $S(Z, A)$ and $a_r(Z)$ when $|A| \in \{1, 2, 3\}$. So

1. If $|A| = 1$ then $A \in \{(0, 0)\}$ and there are only two possible choices for S.

2. If $|A| = 2$ then $|S| = 4$ (there are only five S inferior sets of four elements!) and A may not contain derivatives of a degree higher than three or mixed derivatives (resulting from Pólya condition). Therefore the only possible cases are:

2.1. The case $A = \{(0, 0), (1, 0)\}$ and three possibble choices for P_S : $P_S = span\{1, x, x^2, x^3\}$, $P_S = span\{1, x, x^2, y\}$ and $P_S = span\{1, x, y, xy\}$ to which correspond the following determinants $D(Z, S, A)$: $(x_1-x_2)^4$, $2(x_1-x_2)(y_1-y_2)$ respectively $(y_1 - y_2)^2$.

2.2. $A = \{(0, 0), (1, 0)\}$ and a possible choice for S i.e.

$$P_S = span\{1, x, x^2, x^3\}$$

with $D(Z, S, A) = 12(x_1 - x_2)^2$.

2.3. The other cases are obtained out of the previous ones by interchanging x and y.

So, in this case

$$s(Z, A) \in \{0, 1, 3\}$$

3. The case $|A| = 3$ may be considered similar. Long computation prove that the only possibilites are:

3.1. $A = \{(0, 0), (1, 0), (2, 0)\}$. In this case there are four possible choices for P_S so that $D(Z, S, A)$ is not identically null:

$P_S = span\{1, x, x^2, x^3, x^4, x^5\}, D(Z, S, A) = 4(x_1 - x_2)^9,$
$P_S = span\{1, x, x^2, x^3, x^4, y\}, D(Z, S, A) = 24(x_1 - x_2)^4(y_1 - y_2),$
$P_S = span\{1, x, x^2, x^3, y, xy\}, D(Z, S, A) = 12(x_1 - x_2)(y_1 - y_2)^2,$
$P_S = span\{1, x, x^2, y, xy, x^2y\}, D(Z, S, A) = 4(y_1 - y_2)^3.$
So in this case

$$s(Z, A) = \begin{cases} 4 for x_1 \neq x_2, y_1 \neq y_2 \\ 1, in rest \end{cases}.$$

3.2. $A = \{(0, 0), (0, 1), (3, 0)\}$. Here P_S and the corresponding determinant may be only:
$P_S = span\{1, x, x^2, x^3, x^4, x^5\}, D(Z, S, A) = -288(x_1 - x_2)^7,$
$P_S = span\{1, x, x^2, x^3, x^4, y\}, D(Z, S, A) = -288(x_1 - x_2)^2(y_1 - y_2).$
Therefore

$$s(Z, A) = \begin{cases} 2 for x_1 \neq x_2, y_1 \neq y_2 \\ 1 for y_1 = y_2 \\ 0 for x_1 = x_2 \end{cases}.$$

3.3. A can be one of the following sets: $A = \{(0, 0), (1, 0), (4, 0)\}$, $A = \{(0, 0), (2, 0), (3, 0)\}$, or $A = \{(0, 0), (2, 0), (4, 0)\}$.$P_S$ is only

$$P_S = span\{1, x, x^2, x^3, x^4, x^5\}.D(Z, S, A) = 2880(x_1 - x_2)^5$$

for the first and the second cases, while for the third case $D(Z, S, A) = -34560(x_1 - x_2)^3$.

Therefore, in these cases

$$s(Z, A) = \begin{cases} 1, for x_1 \neq x_2 \\ 0, for x_1 = x_2 \end{cases}.$$

4. $A = \{(0, 0), (1, 0), (0, 2)\}$. In this case P_S and the corresponding determinant are only
$P_S = span\{1, x, x^2, y, y^2, y^3\}, D(Z, S, A) = -24(x_1 - x_2)(y_1 - y_2)^2,$
$P_S = span\{1, x, y, xy, y^2, y^3\}, D(Z, S, A) = 12(y_1 - y_2)^3,$
$P_S = span\{1, x, y, xy, y^2, xy^2\}, D(Z, S, A) = 4(x_1 - x_2)(y_1 - y_2)^2.$

So in this case

$$s(Z, A) = \begin{cases} 3 for x_1 \neq x_2, y_1 \neq y_2 \\ 1 for x_1 = x_2 \\ 0 for y_1 = y_2 \end{cases}$$

5. $A = \{(0,0),(1,0),(1,1)\}$. In this case P_S must be

$$P_S = span\{1, x, x^2, y, xy, x^2y\}$$

and $D(Z,S,A) = -4(x_1 - x_2)^3(y_1 - y_2)$.

Therefore

$$s(Z,A) = \begin{cases} 1 for x_1 \neq x_2, y_1 \neq y_2 \\ 0 in rest \end{cases}$$

6. The other seven cases are similar to the previous ones and result by interchanging x by y.

7. When $A = \{(0,0),(1,0),(0,1)\}$ P_S and the corresponding determinant may be only

$P_S = span\{1, x, x^2, x^3, y, y^2\}, D(Z,S,A) = 2(x_1 - x_2)^4(y_1 - y_2)$,
$P_S = span\{1, x, x^2, x^3, y, xy\}, D(Z,S,A) = (x_1 - x_2)^5$,
$P_S = span\{1, x, x^2, y, xy, x^2y\}, D(Z,S,A) = (x_1 - x_2)^4(y_1 - y_2)$
$P_S = span\{1, x, x^2, y, y^2, y^3\}, D(Z,S,A) = 2(x_1 - x_2)(y_1 - y_2)^4$,
$P_S = span\{1, x, y, xy, y^2, y^3\}, D(Z,S,A) = -(y_1 - y_2)^5$,
$P_S = span\{1, x, y, xy, y^2, xy^2\}, D(Z,S,A) = -(x_1 - x_2)(y_1 - y_2)^4$

and

$$s(Z,A) = \begin{cases} 6 for x_1 \neq x_2, y_1 \neq y_2 \\ 1 for x_1 = x_2, y_1 = y_2 \end{cases}.$$

As a conclusion of these calculations we have:

COROLLARY 1. *For any $Z = \{(x_1, x_2)(y_1, y_2)\}$ we have*

$$a(Z) = t + 4t^2 + 15t^3 + ...$$

for almost all choices of Z (generic Z) and

$$a(Z) = t + 3t^2 + 6t^3 + ...$$

for those Z with $x_1 = x_2$ or $y_1 = y_2$.

COROLLARY 2. *For any $Z = \{(x_1, x_2)(y_1, y_2)\}$ and any A with three elements*

$$a_3(Z) = 15, s(Z,A) = \{1, 2, 3, 4, 6\}.$$

only when $A = \{(0,0),(1,0),(0,1)\}$. *For the nongeneric case (when* $x_1 = x_2$ *or* $y_1 = y_2$*),*

$$a_3(Z) = 6, s(Z, A) = \{0, 1\}.$$

These corollaries show that for rectangular shapes (the above nongeneric case) the numbers $a_r(Z)$ and $s(Z, A)$ tend to be smaller.

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