THE CALCULUS OF A NEW MUSICAL SCALE FREQUENCIES

by
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Abstract: Musical scales were created in time and we no longer have any knowledge about how we came to our current information. At present, due to computers and having particularly capable instruments at our disposal, we can moot the question of scale recalculation and the creation of new, computer-generated scales. Likewise, computers allow the creation of synthesising instruments, which can be diatonically tuned in all the scales, an impossible thing to achieve in the case of classical instruments.

The way in which musical scales were created is wrapped up in the dark of time. A sure thing is that the first constructors of musical instruments used as a work basis the sound phrase and not the frequency. This is obvious through the fact that they did not have the possibility to measure frequency. If they used the monochord (single-stringed instrument), the sound phrase is directly proportional to the length of the string. (It also depends on other parameters, but for a certain monochord adjustment, these parameters were constant).

The Pythagoreans discovered and studied the most perfect musical proportion, defined by the series of numbers: $a, \frac{a+b}{2}, \frac{2ab}{a+b}, b$. While calculating we consider the different lengths of the monochord (therefore the sinusoidal signal period). From mathematics (the relations between the arithmetic, geometric and melodic means) we know that the following relation is true: $a > \frac{a+b}{2} > \frac{2ab}{a+b} > b$. Pythagoras used this proportion to establish the diatonic scale that bears his name. For a=2 and b=1, we have the series of numbers 2, 3/2, 4/3, 1.

The same proportions can be obtained by using sound frequency (the frequency is proportional to the inverse of the period). 1, 4/3, 3/2, 2.

Therefore, in order to obtain harmony, the interval of an octave requires an unequal division, by the above proportion. The unequal intervals create big problems when tuning musical instruments in various scales. If we want a musical instrument to play in more than one scale correctly, then the division of an octave in equal parts is necessary. That is, the interval between two adjacent notes should be the same irrespective of the notes.

If we want to divide the octaves in two equal intervals then, (considering that the frequency of the note from half interval is a fractional number) we obtain the relation: $\frac{1}{\frac{x}{y}} = \frac{\frac{x}{y}}{2}$. From this follows $\frac{x^2}{y^2} = 2$ or $\frac{x}{y} = \sqrt{2}$. Unfortunately, the division of the octave in equal parts does not produce harmony but dissonance. A musical scale (a compromise between the two), which would combine the advantages of each of these is needed. That is how the temperate scales appeared.

For the C major scale the relation 1, 4/3, 3/2,2, represents the Notes C – F – G– C_2. The notes that have the relation between the frequencies as a multiple of 2 bear the same name with a different numerical index, depending on which scale they are in.

In order to construct the diatonic scale we have to do something else too. We start from the premises that the notes whose frequency is in a relation $\frac{fX}{fY} = \frac{n}{m}, \textit{unde } m \textit{ si } n \leq 6$ (fX and fY are the frequencies of notes X and Y) create harmony.

In these conditions the major chord of the diatonic scale is built on three notes that have the frequency relation proportional with the numbers 4, 5 and 6. In the case of the diatonic scale C major, the notes corresponding to the frequencies 4, 5 and 6 are C, D and G.

The diatonic scale is constructed in such a way as to create two more major chords. Therefore, the complete diatonic scale contains three major chords. These are constructed as follows: C - E - G, a - b - C_2 - a_2 and G - c - d - G_2.

Why there are three major scales and why they are constructed like this is one of the mysteries of music and these data were obtained due to the genius of those who created musical scales. This material tries to find the musical scales mathematically.

In order to solve the problem we make some specifications and simplifications.

- First, we work with frequencies;
- Second, we consider the frequency of the note fC=1. Hence the frequencies: fE = 4/3, fG = 3/2and $fC_2 = 2$.

From the second major chord (a - b - C_2 - a_2) results:

$\frac{a}{4} = \frac{b}{5} = \frac{2}{6} = \frac{1}{3}$. Hence: $a = \frac{4}{3} = 1,(3)$ and $b = \frac{1}{3} \times 5 = \frac{5}{3} = 1,(6)$.

These frequencies correspond to the notes F and A.

From the third major accord (G- c - d - G_2) results:

$\frac{\frac{3}{2}}{4} = \frac{c}{5} = \frac{d}{6} = \frac{3}{8}$. Hence: $c = \frac{3}{8} \times 5 = \frac{15}{6} = 1{,}875$ and

$d = \frac{3}{8} \times 6 = \frac{18}{8} = 2{,}25$.

These frequencies correspond to the notes B and D_2. The frequency of D corresponds to the frequency of D_2 / 2, thus 2,25/2 = 1,125.

Coming back to the real frequencies we shall calculate for the frequency of the note A=440 Hz. Results the table below.

Name of the note	C	D	E	F	G	A	B	C2	D2
Relative frequencies	1	1.125	1.25	1.333	1.5	1.667	1.875	2	2.25
The first major chord	4		5		6			8	
The second major chord				4		5		6	
The third major chord		3			4		5		6
Absolute frequencies	264	297	330	352	396	440	493	528	594

Table 1.

Extremely important for musical effect is the interval. The interval between two notes is defined as the ratio between the corresponding note frequencies. Thus, the interval C – G is defined as the ratio 1,5 / 1 = 1,5 = 3/2 and is called perfect fifth. The interval G – C_2 is defined as the ratio 2 / 1,5 = 1,(3) = 4/3 and is called perfect fourth. According to Pythagors's scale, the intervals C – F and G – C_2 are perfect fourths, and the intervals C – G and F – C_2 are perfect fifths.

The sum of two intervals is calculated by multiplying the values of the corresponding intervals. For instance: the sum of C – G and G – C_2 is calculated by multiplying 1,5 x 1,(3) = 2. The interval C – C_2 calculated from the frequency ratio is also 2 (2 / 1 = 2).

The table above must be completed with the interval between the adjacent notes as follows:

Name of the note	C	D	E	F	G	A	B	C2
Relative frequencies	1	1,125	1,25	1,(3)	1,5	1,(6)	1,875	2
Intervals		9/8	10/9	16/15	9/8	10/9	9/8	16/15

Table 2.

We notice that there are intervals equal to 9/8 or 10/9 called tones and intervals equal to 16/15 called semitones. The problem appears because of the fact that the tones are of two values 9/8 and 10/9. In order to correct this difference all sorts of compromises were made, resulting in the commas, which represent the various differences between these frequencies.

With an error of approximately 1%, the temperate scales were introduced. The temperate scales divide an octave in 12 equal intervals called semitones (they are only approximately equal to the semitones of the diatonic scale). Two adjoining semitones form a tone.

Starting from the observation that the sum of two intervals is obtained by multiplying their values and that the one octave interval is made of 12 semitones, the value of a semitone is obtained as follows:

$$Semitone = \sqrt[12]{2} = 1{,}0594630943592952645618252949463 4$$

In order to complete the frequencies in the new C scale called temperate, we complete the following table:

Name of the note	C	D	E	F	G	A	B	C2
Relative frequencies	1.000	1.122	1.260	1.335	1.498	1.682	1.888	2.000
The first major chord	261.6	293.7	329.6	349.2	392.0	440.0	493.9	523.3

Table 3.

Let us try and obtain the temperate scale mathematically, using the information above.

We start from the major accord, which has a ratio of frequencies 4, 5 and 6. Follows C_2 corresponding to 8. The corresponding intervals are obtained by multiplying the frequencies:

C - E, corresponding to fE/fC = 5/4;
E - G, corresponding to fG/fE = 6/5;
G – C_2, corresponding to fC2/fG = 8/6 = 4/3.

Results the equation system:

$$\begin{cases} 6/5 = x^{n_1} \\ 5/4 = x^{n_2} \\ 4/3 = x^{n_3} \\ x = \sqrt[m]{2} \\ n_1 + n_2 + n_3 = m \end{cases}$$

We notice that $n_1 < n_2 < n_3$. But $n_1 + n_2 + n_3 = m$. As we want m to be as small as possible in order to create the most accessible scale, we set the condition that the inequality be fulfilled at the limit, that is n_1, n_2 and n_3 should differ by one unit. We shall write the new system:

$$\begin{cases} 6/5 = x^{n} \\ 5/4 = x^{n+1} \\ 4/3 = x^{n+2} \\ x = \sqrt[m]{2} \\ n + (n+1) + (n+2) = m \end{cases}$$

From the first three equations we calculate x:

$$x \cong \frac{5/4}{6/5} = 5/4 \times 5/6 = \frac{25}{24} = 1{,}04166666667$$

$$x \cong \frac{4/3}{5/4} = 4/3 \times 4/5 = \frac{16}{15} = 1{,}06666666667$$

For a better approximation we calculate the geometrical mean between the two values:

$$x \cong \sqrt{x^2} = \sqrt{\frac{4/3}{6/5}} = \sqrt{4/3 \times 5/6} = \sqrt{20/18} = \sqrt{1{,}11111111111} = 1{,}05409255339$$

Replacing x in the penultimate equation, we obtain:

$$1{,}05409255339 = \sqrt[m]{2}$$

Results:

$$m = \frac{\lg(2)}{\lg(1{,}05409255339)} = \frac{0{,}30102999566}{0{,}02287874528} = 13{,}1576269579$$

Consequent to this calculus results that the octave should be divided into 13 equal intervals. In this way, the error can be calculated as follows:

$$1{,}05409255339^{13{,}1576269579} = 2{,}00000000001125634777140219127741$$

$$\varepsilon = \frac{2{,}00000000001125634777140219127741 - 2}{2} = 5{,}62817388570 \times 10^{-12}$$

The relative error is extremely small, but we are not pleased with a division into 13 intervals, as 13 is a prime number. We try with 12, which has a multitude of divisors.

$$x = \sqrt[12]{2} = 1{,}05946309436$$

This value is different from 1,05409255 previously calculated just at the fourth digit. The relative error is in this case:

$$\varepsilon = \frac{1{,}05409255339 - 1{,}05946309436}{1{,}05409255339} = -0{,}00509494252$$

So, around 0.5% it is acceptable enough for the human ear.

The conclusion is that by dividing the octave in 12 equal intervals we obtain a scale that approximates the diatonic scale, having in relation with it a difference of approximately 0.5% acceptable for the common musical hearing. For specialists, who sense the difference, it is recommended that the instrument be tuned in the diatonic scale, with the specification that this one is very hard to use as it does not have equal intervals. The new scale obtained is the temperate scale.

References:

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