THE HAROS-FAREY SEQUENCE AT TWO HUNDRED YEARS
A SURVEY

by
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Abstract. Let $\mathfrak{F}_Q$ be the set of representatives for the nonnegative subunitary rational numbers in their lowest terms with denominators at most Q and arranged in ascending order. This finite sequence of fractions has two remarkable basic properties. The first one asserts that the difference between two consecutive fractions equals the inverse of the product of their denominators. The second, called also *the mediant property*, says that if a'/q', a''/q'' and a'''/q''' are consecutive in $\mathfrak{F}_Q$ then $a''/q'' = (a' + a''')/(q' + q''')$. These properties are equivalent and they were mentioned without proof for the first time by Haros in 1802 and respectively by Farey in 1816, independently. Thus the proper name for $\mathfrak{F}_Q$ should be "the Haros-Farey sequence" instead of "the Farey sequence" as it is known after Cauchy.

Besides marking the two centuries anniversary of the Farey sequence, the main raison d'être of this article is to survey some important properties of $\mathfrak{F}_Q$, most of them discovered recently. We also sketch the impact of these results on different problems of Number Theory or Mathematical Physics. There are many papers (more than five hundred published only in the last fifty years) dealing with $\mathfrak{F}_Q$, and here we only mention a few of them. Though, starting with the cited articles below, one may easily track most of the remaining ones.

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1. Introduction and historical notes

Since the ancient Egypt era, from which the Ahmes (Rhind) papyrus is one of the oldest written pieces of mathematics which came down to us, fractions are widely used in mathematics. They raise many problems, of which, even today, some remain as difficult as they have ever been. A common fraction a/q can be viewed as a relation between two integers a and $q \neq 0$, and the set of all such fractions preserves all the mystery the integers have. The Egyptians worked only with *unit fractions* (fractions with numerator equal to one, known also as *Egyptian fractions*) and their problem was to write any given common fraction as a sum of different unit fractions. Farey fractions may be used as a tool to solve Egyptian's task and there are many algorithms which produce short representations, and also representations with small denominators (see Bleicher [Ble1972]).

The way in which the unit fractions are positioned with respect to one another is well understood, although this still contains a wealth of information about numbers, which inspired John Napier [Nap1614] four hundred years ago and led to the very important discovery of the natural logarithm. The distribution of rationals in $[0,1]$ with bounded denominator raises difficult and interesting questions.

Here, we consider all the reduced subunitary fractions with denominators bounded by a fixed margin. More precisely, let Q be a positive integer and denote by $\mathfrak{F}_Q$ the set of irreducible fractions in $[0,1]$ whose denominator does not exceed Q, that is

$$\mathfrak{F}_Q = \left\{ \frac{a}{q}\colon\ 0 \le a \le q \le Q,\ (a,q)=1 \right\}.$$

The cardinality of $\mathfrak{F}_Q$ is

$$N(Q) = \#\mathfrak{F}_Q = 1 + \sum_{n\le Q} \varphi(n) = \frac{3}{\pi^2}Q^2 + O(Q\log Q)\,.$$

We assume that the elements of $\mathfrak{F}_Q$ are arranged increasingly and for any n with $1 \le n \le N(Q)$, we write the fractions as $\gamma_n = \frac{a_n}{q_n}$, in which $(a_n, q_n) = 1$ and $0 \le a_n \le q_n \le Q$. Notice the symmetry of $\mathfrak{F}_Q$ with respect of $1/2$. For example when $Q = 7$, we have

$$\mathcal{F}_7 = \left\{ \tfrac{0}{1}, \tfrac{1}{7}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{6}{7}, \tfrac{1}{1} \right\}.$$

The sequence $\mathfrak{F}_Q$ is known as the Farey sequence (series) of order Q, but during the last century a number of authors questioned this label. Farey was a geologist who published a note [Far1816e] in which he observes the *"mediant property"* of $\mathfrak{F}_Q$. This states that if $a/q < a'/q' < a''/q''$ are consecutive elements of $\mathfrak{F}_Q$ then

$$\frac{a'}{q'} = \frac{a+a''}{q+q''}\,. \tag{1.1}$$

Reading a French translation of the note [Far1816f], in the same year, 1816, Cauchy (see [Cau1840, Vol. I, 114–116]) proves the "curios property", as Farey called it, and includes the proof in an earlier edition of his *Exercices de mathématique*. But this is not the first time when $\mathfrak{F}_Q$ appears on a published paper. Two hundred years ago, in 1802, C. Haros [Har1802] showed how one could construct $\mathcal{F}_{99}$ and uses the mediant property in the process. In fact Haros used only the fact that the mediant is between the fractions that defines it. Although, he explicitly stated that the difference between consecutive fractions in $\mathcal{F}_{99}$ is one over the product of their denominators, that is relation (1.2) below. A largely quoted sentence of Hardy [Har1959] attributes "the Farey's immortality" to his “failure to understand a theorem which Haros had proved perfectly fourteen years before.” This is an overstatement, since Haros’s construction does not qualify as a proof and there is no way to know whether Farey new about Haros’s paper (in fact Farey says at the end of [Far1816e] that he “is not acquainted, whether this curious property of vulgar fractions has been before pointed out; or whether it may admit of some easy or general demonstration”). We remark the role played by tables in the discoveries made by both Haros and Farey. More comments on the subject have been made, for example, by Dickson [Dic1938, Vol.1, pag. 156], Hardy and Wright [HW1979]. Bruckheimer and A. Arcavi [BA1995] give more details on the texts that propagated the error until this day.

A natural quadratic generalization of the Farey sequences was introduced and studied numerically by Brown and Mahler [BM1971]. Afterwards, trying to make some justice to Haros, Delmer and Deshouillers [DD1993], [DD1995] have called this new sequence *the Haros (or quadratic) sequence*. Taking all these into account, after two hundred years one may agree that the appropriate name for what is largely known as “the Farey sequence” should be “the Haros-Farey sequence”. But still, even in the present survey, we are forced by the multitude of articles on the subject to keep in use the historical name.

There is a second basic property of $\mathfrak{F}_Q$, equivalent to (1.1), which characterizes consecutive fractions of $\mathfrak{F}_Q$. This says that

$$\Delta(\gamma_j, \gamma_{j+1}) = 1, \quad \text{for any } 1 \leq j \leq N(Q) - 1\,, \tag{1.2}$$

in which $\Delta(\gamma_i, \gamma_j) = \Delta(i,j) = -\left|\begin{smallmatrix} a_i & a_j \\ q_i & q_j \end{smallmatrix}\right|$, for any $\gamma_i = a_i/q_i$ and $\gamma_j = a_j/q_j$ in $\mathfrak{F}_Q$.

Different geometric interpretations of the Farey sequence show their usefulness in different contexts. One of them is through Ford’s circles (see [For1938] and [Max1985]). But the first interpretation one could imagine is to represent each fraction a/q with $0 \leq a \leq q \leq Q$ as a lattice point (q,a) in the cartesian plane. This was the source of inspiration to an ingenious proof of (1.1) discovered by Sylvester. We call a fraction a/q visible if the segment of the straight line connecting the origin with the point (q,a) contains no other lattice points. Now if we send a ray from the origin along the x-axis and then rotate it counterclockwise, it is clear that the ray will end up only in points with $(a,q) = 1$. This means that we have a perfect correspondence between the fractions from $\mathfrak{F}_Q$ and the set of visible points situated in the triangle with vertices $(0,0)$, $(Q,0)$, (Q,Q). Moreover, each fraction equals the slope of the line connecting the origin to the corresponding point, and the ray touches the points successively in an order that agrees with the ascending order of $\mathfrak{F}_Q$. In particular, if $\gamma' = a'/q'$ and $\gamma'' = a''/q''$ are consecutive Farey fractions, than the triangle with vertices $(0,0)$, (q',a'), (q'',a'') contains no other lattice points inside or on the edges. The area of this triangle equals $1/2$. (Sylvester observed this directly, but afterwards, in 1899, Pick discovered a more general statement that says: "The area of a polygon whose vertices are lattice points equals the number of points inside plus a half of the number of points on the boundary minus one.") On the other hand, from analytic geometry we know that the area of the triangle is also equal to $\Delta(\gamma', \gamma'')/2$. This proves (1.2), and (1.1) follows immediately from it. There are many other different arguments to prove the two equivalent basic properties, three of them presented by Hardy and Wright [HW1979, Chapter III].

We remark that relations (1.1) and (1.2) allow us to determine recursively all the elements of $\mathfrak{F}_Q$. Thus, starting with two "parent fractions" from $\mathfrak{F}_Q$, one can insert successively all the mediants with denominators at most Q to get all the Farey fractions in that interval. In particular starting with $0/1$ and $1/1$, one obtains all the elements of $\mathfrak{F}_Q$. On the other hand, (1.2) produces the fractions in a row increasingly (or decreasingly). We see that for any two relatively prime integers $0 \leq q_1,\, q_2 \leq Q$ there exists exactly one pair of consecutive fractions in $\mathfrak{F}_Q$ with denominators q_1, q_2. Also, a neighbor denominator is uniquely determined by q_1 and q_2 (for more details see Section 4).

Farey fractions are often better suited than regular continued fractions in all kind of approximation problems. They are a useful tool in a variety of domains, especially in the circle method (started by Hardy and Littlewood in the early 1920's and significantly enhanced over the years, see [DFI1994]) and in the rational approximation to irrationals (see the head-stone paper of Hurwitz [Hur1894]). The proof of a nice recent result of Aliev and Zhigljavsky [AZ1999] requires approximations with Farey fractions. They determine, for any given irrational number $\theta \in (0,1)$, the two-dimensional asymptotic distribution of the pairs $\big(n \min x_k,\, n(1-\max x_k)\big)$, with $1 \leq k \leq n$, as $n \to \infty$, where $x_k := k\theta \pmod 1$ are the elements of the Weyl sequence of order n. (A remarkable fact says that for any n, the interval $[0,1]$ is partitioned $\pmod 1$ by the Weyl sequence in subintervals of only two or three distinct lengths.) Also, many sieve methods appeal to Farey fractions and almost any application of the large sieve to number theory starts with a sum over the elements of $\mathfrak{F}_Q$ (see Montgomery [Mon1978, § 8]).

Generalizations of the real Farey fractions to the complex plane were tried by Cassels, Ledermann and Mahler [CLM1951] and by Schmidt [Sch1969]. Cassels, Ledermann and Mahler introduced and studied the so-called *Farey sections* for $\mathbb{Q}(i\sqrt{m})$ with $m = 1, 3$ (see also Leveque [LeV1952]). Schmidt's approach is along different lines. He generalizes the *Farey interval* (distance between two consecutive Farey fractions) to *Farey triangles* and *Farey quadrangles*. This applies only for a few quadratic fields, $\mathbb{Q}(i\sqrt{m})$ with $m = 1, 2, 3, 7$. Schmidt uses his construction to investigate the approximation spectra of the corresponding fields. (In the case of $\mathbb{Q}(i\sqrt{m})$, the approximation spectra is the set of all constants $C(\xi)$, where $C(\xi)$ for any $\xi \notin \mathbb{Q}(i\sqrt{m})$ is defined as $C(\xi) = \limsup(|a|\,|a\xi - b|)^{-1}$, the $\limsup$ being taken over all algebraic integers $a, b \in \mathbb{Q}(i\sqrt{m})$, $a \neq 0$.) Building along the same ideas is the work of Moeckel [Moe1982] and the concept of *Farey tesselations* of Vulakh [Vul1999].

There are many more papers dealing with Farey fractions and the ones presented here may serve as good starting points. We end this section by mentioning one more work of Plagne [Pla1999], who uses Farey fractions as a tool to prove a uniform version of Jarník's theorem, which states that for any given function $f(x)$ tending to infinity there exists a strictly convex curve $\mathcal{C}$ and a strictly increasing sequence of integers $(q_n)_{n\geq 0}$, such that for each n, $\left|\mathcal{C} \cap (\frac{1}{q_n}\mathbb{Z})^2\right| \geq \frac{q_n^{2/3}}{f(q_n)}$.

2. **Farey fractions and the Riemann Hypothesis**

The first one who noticed the connection between the distribution of $\mathfrak{F}_Q$ and the Riemann hypothesis (to shorten, RH from now on) was Franel [Fra1924]. His result was then clarified by Landau [Lan1924] and [Lan1927]. For a different, more elementary approach, see Zulauf [Zul1977].

The Farey fractions are very nicely distributed in $[0,1]$. Before the computer era, this even made Neville [Nev1950] to compile a book with over four hundred pages containing the 319765 elements of $\mathcal{F}_{1025}$. One way to measure the departure of $\mathfrak{F}_Q$ from a perfectly uniformly distributed sequence is to find the displacements $\delta_j = \gamma_j - \frac{j}{N(Q)}$ and to show that in average they are small. But this is not easy at all, since Franel proved the equivalence

$$\text{RH} \iff \sum_{j=1}^{N(Q)} |\delta_j| = O\big(Q^{1/2+\epsilon}\big)\,. \tag{2.1}$$

In fact he showed that the estimate is equivalent to another real line statement of RH, which says that $\sum_{n\leq x}\mu(n) = O(Q^{1/2+\epsilon})$, where $\mu(n)$ is the Möbius function. Landau gives a similar version showing that

$$\text{RH} \iff \sum_{j=1}^{N(Q)} \delta_j^2 = O\big(Q^{-1+\epsilon}\big)\,.$$

On the lower bound side Stechkin [Ste1997] showed unconditionally that

$$\sum_{j=1}^{N(Q)} \delta_j^2 \gg Q^{-1}\log Q\,,$$

disproving a conjecture of Sato, who guessed that the lower bound should be no more than $Q^{-1}(\log\log\log Q)^2$. Answering a question of Davenport, Huxley [Hux1971] proved a result for Dirichlet L-functions which is similar to that of Franel.

Subsequently different authors proved that other statements of this sort are equivalent to RH. Some interesting examples are:

$$\mathrm{RH} \iff \sum_{j=1}^{N(Q)} \Big(\gamma_j^2 - \frac{1}{3}\Big) = O\big(Q^{1/2+\epsilon}\big)\,,$$

$$\mathrm{RH} \iff \sum_{j=1}^{N(Q)} \Big(\gamma_j^3 - \frac{1}{4}\Big) = O\big(Q^{1/2+\epsilon}\big)\,,$$

which are due to Kopriva, Mikolás and Zulauf, or the asymmetric ones:

$$\mathrm{RH} \iff \sum_{j=1}^{[N(Q)/2]} \Big(\gamma_j - \frac{1}{4}\Big) = O\big(Q^{1/2+\epsilon}\big)\,,$$

$$\mathrm{RH} \iff \max_{0\le t\le 1} \Big|\sum_{\gamma_j\le t} \delta_j\Big| = O\big(Q^{1/2+\epsilon}\big)\,,$$

which are stated by Zulauf and respectively by Kanemitsu and Yoshimoto. Using arithmetical considerations on Dirichlet characters and L-functions, in [KY1997] were established other "short-interval" results, that is $\frac{1}{5}$, $\frac{1}{6}$-results. More generally, Kanemitsu and Yoshimoto [KY1996] showed that for any function f belonging to a large class of so called *Kubert functions*, the following equivalence holds:

$$\mathrm{RH} \iff \sum_{j=1}^{N(Q)} f(\gamma_j) = O\big(Q^{1/2+\epsilon}\big)\,.$$

For a general account on the connections between RH and the Farey fractions see [KY1996], [Yos1998], [KY1997], [Yos2000] and the references therein.

Another way to see if a sequence is nicely distributed is to see if its discrepancy is small. The *discrepancy* of $\mathfrak{F}_Q$ is defined as

$$D_Q := \sup_{0\le\alpha\le 1} \left| \frac{\#(\mathfrak{F}_Q \cap [0,\alpha])}{N(Q)} - \alpha \right| .$$

Improving on an earlier result of Neville, Niederreiter [Nie1973] has shown that for any α with $0 \le \alpha \le 1$, we have $D_Q \ll (\#\mathfrak{F}_Q)^{-1/2} \ll Q^{-1}$. Later Dress [Dre1999] proves the unexpected equality $D_Q = Q^{-1}$.

In fact, we already knew that $\mathfrak{F}_Q$ is uniformly distributed $\pmod 1$, since for any Riemann integrable function f defined on $[0,1]$, one has

$$\lim_{Q\to\infty} \frac{1}{N(Q)} \sum_{j=1}^{N(Q)} f(\gamma_j) = \int_0^1 f(t)\,dt\,,$$

verifying Weil's criterion, as Mikolás [Mik1949] has proved. This inspired Koch to ask for which functions $f \in L^1[0,1]$ is the Riemann hypothesis equivalent to the estimate $E_f(x,Q) = O(Q^{1/2+\epsilon})$. Here $E_f(x,Q)$ is the shorter interval Koch-Mikolás remainder:

$$E_f(x,Q) = \sum_{\gamma_j \le x} f(\gamma_j) - N(Q) \int_0^x f(t)\,dt\,,$$

The point here is that knowing good bounds for $E_f(x,Q)$ for certain classes of test functions f may lead to some insight in understanding RH. This problem was considered, for example, by Codecà [Cod1981], Codecà and Perelli [CP1988], Yoshimoto and others in a series of papers (see [Yos1998], [Yos1998]).

3. The distribution of spacings between Farey points

It is generally accepted that the Farey sequence is uniformly distributed in $[0,1]$, but it is rather hard to measure the size of this uniformity. One believes that there might be some correspondence between consecutive Farey points and differences between consecutive zeros of $\zeta(s)$. What is presently known is that if there is such a correspondence, it is not a trivial one, because these two sequences have different distributions.

Since the discovery in the early 1970's of the fact that the zeros of the Riemann zeta function and the eigenvalues of GUE matrices have the same pair correlation function (the GUE hypothesis or the Mongomery-Odlyzko Law), much effort has been put by number theorists trying to develop techniques to prove the conjecture. Following the principle that says to try first the analogue of an inabordable problem in different settings and along the way to learn techniques that might prove useful, over the years different authors studied the distribution of a number of sequences, such as the prime numbers (under the h-tuple conjecture), the values of the Kloosterman sums, the primitive roots mod p, the set of fractional parts $\{\alpha f(n)\}$, where α is real and $f(x)$ is a polynomial with integer coefficients, the Montgomery-Odlyzko Law for some classes of zeta and L-functions over finite fields, etc.

As in music, where simple notes are not interesting in themselves and intervals between them create the melody, spacings between the elements of a sequence are those who determine the distribution. There is no general best concept that measures the distribution of a sequence, but two ways to proceed are widely accepted. One of them is to obtain the m-level correlation measure for any $m \geq 2$, while the other asks for the h-th level consecutive spacing measure for any $h \geq 2$. The basic properties (1.1) and (1.2) make the second approach to be more convenient for $\mathfrak{F}_Q$.

3.1. **Notations.** Most of the statements we present about the distribution of the Farey sequence apply for the fractions in a subinterval of $[0, 1]$. This leads us to introduce some appropriate notations. A basic property of the Farey sequence is a type of heritage property. This manifests mainly when Q gets large. Then, on average, the elements of $\mathfrak{F}_Q$ relate to one another on short intervals in the same way as on the complete interval $[0, 1]$, and as a consequence, the corresponding distribution functions are the same. In general this can be proved by bringing into play the "Kloosterman machinery", as the authors first realized during the Christmas holiday of 1996 at a meeting at the University of Rochester. The basic idea is to write the condition $\gamma_j \in I$ in terms of denominators only. This is achieved by observing that the equality $\gamma_j - \gamma_{j-1} = a_j/q_j - a_{j-1}/q_{j-1} = 1/(q_{j-1}q_j)$ implies $a_j = \bar{q}_{j-1} \pmod{q_j}$. Then, if I is a subinterval of $[0, 1]$, the statement $\gamma_j \in I$ is equivalent to $\bar{q}_{j-1} \in q_j I$, as needed.

Let $I = [\alpha, \beta]$ be a subinterval of $[0, 1]$ and denote by $\mathfrak{F}_Q(I) := \mathfrak{F}_Q \cap I$ the set of Farey fractions of order Q from I. The number of elements of $\mathfrak{F}_Q(I)$ is

$$N(Q, I) = |I| \cdot N(Q) + O(Q \log Q) = 3|I|Q^2/\pi^2 + O(Q \log Q).$$

A fundamental geometrical interpretation of $\mathfrak{F}_Q$ is through the set of lattice points with relatively prime coordinates in the triangle with vertices $(Q, 0)$, (Q, Q) and $(0, Q)$. These points are in correspondence with the set of pairs of consecutive denominators of fractions from $\mathfrak{F}_Q$. By down-scaling by a factor of Q, we get $\mathcal{T}$, the so called *Farey triangle*. This is defined by

$$\mathcal{T} = \left\{(x, y)\,;\, 0 < x \leq 1,\, 0 < y \leq 1,\, x + y > 1\right\}.$$

Further, keeping the new scale and looking at consecutive denominators, we consider for each $(x, y) \in \mathbb{R}^2$, the sequence $\left\{L_i(x, y)\right\}_{i \geq 0}$ defined by $L_0(x, y) = x$, $L_1(x, y) = y$ and then recursively, for $i \geq 2$,

$$L_i(x, y) = \left[\frac{1 + L_{i-2}(x, y)}{L_{i-1}(x, y)}\right] L_{i-1}(x, y) - L_{i-2}(x, y)\,. \tag{3.1}$$

For each cube $\mathcal{C} = (\alpha_1, \beta_1) \times \cdots \times (\alpha_h, \beta_h) \subset (0, \infty)^h$, let $\widetilde{\Omega}_{\mathcal{C}}$ be given by:

$$\widetilde{\Omega}_{\mathcal{C}} = \bigcap_{i=1}^{h} \left\{ (x, y) \in \mathcal{T} : \ \frac{3}{\pi^2 \beta_i} < L_{i-1}(x, y) L_i(x, y) < \frac{3}{\pi^2 \alpha_i} \right\}. \tag{3.2}$$

3.2. **The h-spacing distribution.** To define the h-spacing distribution of a sequence, one must first apply a standard normalization to the sequence in order to get a measure suitable to be compared with those attached to other sequences. Thus, we suppose that $x_0 \leq x_1 \leq \cdots \leq x_N$ are N given real numbers with mean spacing about 1. Then *the h^{th} level consecutive spacing (probability) measure ν_h* is defined on $[0, \infty)^h$ by

$$\int\limits_{[0,\infty)^h} f \, d\nu_h = \frac{1}{N-h} \sum_{j=1}^{N-h} f\left(x_{j+1} - x_j, x_{j+2} - x_{j+1}, \ldots, x_{j+h} - x_{j+h-1}\right),$$

for any $f \in C_c\big([0, \infty)^h\big)$.

More precise information on a sequence is known if one gets the h-level of the intervals[1] distribution of a sequence. For any integer $d \geq 1$, *the h^{th} level of the d intervals probability ν_h^d* is defined on $[0, \infty)^h$ similarly by

$$\int\limits_{[0,\infty)^h} f \, d\nu_h^d = \frac{1}{N-h} \sum_{j=1}^{N-h} f\left(x_{j+d+1} - x_j, x_{j+d+2} - x_{j+1}, \ldots, x_{j+d+h} - x_{j+h-1}\right),$$

for any $f \in C_c\big([0, \infty)^h\big)$. One should notice that $\nu_h^1 = \nu_h$.

3.3. **The distribution μ_h.** Following the general rule to get the h-spacing distribution, in the particular case of $\mathfrak{F}_Q$, we first normalize the sequence $\mathfrak{F}_Q(I)$ and put $x_n = N(Q, I)\gamma_n/|I|$ to get, for each Q, the sequence $\{x_n\}_{1 \leq n \leq N(Q,I)}$ with mean spacing unity. Correspondingly, we obtain a sequence $\big\{\mu_Q^{h,I}\big\}_{Q \geq 1}$ of probability measures on $[0, \infty)^h$. The convergence of this sequence assures the existence of the h-spacing distribution of $\mathfrak{F}_Q$. This was proved by Augustin, Boca, Cobeli and Zaharescu in [ABCZ2001]. They showed that the sequence $\big\{\mu_Q^{h,I}\big\}_{Q \geq 1}$ converges weakly to a probability measure μ_h, which is independent of I. The repartition of μ_h is given by

$$\mu_h(\mathcal{C}) = 2\text{Area}(\widetilde{\Omega}_{\mathcal{C}}), \qquad \text{for any box } \mathcal{C} \subset (0, \infty)^h.$$

[1] We use the word interval also with a meaning as in the intervollic theory from music. Here the spacing $x_{j+1} - x_j$ determines an interval of a second, $x_{j+2} - x_j$ determines an interval of a third, $x_{j+3} - x_j$ determines an interval of a fourth, and so on.

The case $h = 1$ and $I = [0,1]$ has been considered by Hall [Hal1970] and later Delange [Del1974] generalized the result for shorter intervals. Tâm [Tam1974] treated the bidimensional case showing that the the pairs $Q^2(\gamma_{j+1}-\gamma_j, \gamma_{j+2}-\gamma_{j+1})$, have a limit distribution as Q tends to infinity.

The repartition function of μ_1 is given by

$$
\begin{aligned}
G_1(t) &= \int_t^\infty d\mu_1(x) = 1 - 2\mathrm{Area}\left(\left\{(x,y)\in\mathfrak{T};\, xy > 3/(\pi^2 t)\right\}\right)\\
&= \begin{cases}
1, & \text{for } 0 \le t \le \dfrac{3}{\pi^2},\\
-1 + \dfrac{6}{\pi^2 t} - \dfrac{6}{\pi^2 t}\log\dfrac{3}{\pi^2 t}, & \text{for } \dfrac{3}{\pi^2} \le t \le \dfrac{12}{\pi^2},\\
-1 + \dfrac{6}{\pi^2 t} + \sqrt{1-\dfrac{12}{\pi^2 t}} - \dfrac{12}{\pi^2 t}\log\dfrac{1+\sqrt{1-\frac{12}{\pi^2 t}}}{2}, & \text{for } \dfrac{12}{\pi^2} \le t.
\end{cases}
\end{aligned}
\tag{3.3}
$$

This shows that μ_1 is absolutely continuous with respect to the Lebesgue measure on $[0,\infty)$. The density of μ_1, denoted by $g_1(t)$, has different formulae on each of the three intervals from (3.3). This is

$$
g_1(t) = \begin{cases}
0, & \text{for } 0 \le t \le \dfrac{3}{\pi^2},\\
\dfrac{6}{\pi^2 t^2}\log\left(\dfrac{\pi^2 t}{3}\right), & \text{for } \dfrac{3}{\pi^2} \le t \le \dfrac{12}{\pi^2},\\
\dfrac{12}{\pi^2 t^2}\log\left(\dfrac{\pi^2 t}{6}\left(1-\sqrt{1-\dfrac{12}{\pi^2 t}}\right)\right), & \text{for } \dfrac{12}{\pi^2} \le t,
\end{cases}
\tag{3.4}
$$

and the graph of $g_1(t)$ is shown in Figure 3.1. The fact that $g_1(t)$ vanishes on an entire interval to the right of the origin means that if we were sitting at a Farey point it is extremely unlikely that we will find another point close by.

Let us see where this stands on the larger picture that concerns the distribution of numerical sequences. For randomly distributed numbers–the Poissonian case–, the h-level consecutive spacing limiting measure μ_h is $d\mu_h(\lambda_1,\dots,\lambda_h) = e^{-(\lambda_1+\cdots+\lambda_h)}d\lambda_1\dots d\lambda_h$. By (3.3) one finds that the proportion of differences between consecutive elements of $\mathfrak{F}_Q$ that are larger than the average equals $G_1(1) = 6\left(1-\log\left(3/\pi^2\right)\right)/\pi^2 - 1 = 0.3318\dots$ as $Q\to\infty$, which is smaller than the value $1/e = 0.36787\dots$, expected if the Farey fractions were placed in $[0,1]$ as a result of a Poisson process. The shape of the density (3.4) places the Farey sequence at one end, a Poissonian distributed sequence at the other end, and somewhere in the middle the statistical model of Random Matrix Theory that corresponds to GUE. In this last case, the density of the nearest neighbor distribution of the bulk spectrum of random matrices–known as *the Gaudin density*–has no closed form, but can be computed numerically. In Figure 3.2 the Gaudin density is compared to the Poissonian density with mean 1. One can see that in the Poissonian case, small spacings are quite probable, they are rare in the GUE case, while in the case of $\mathfrak{F}_Q$ they are completely missing. Thus one can say that the eigenvalues repel one another in the GUE case, and that each Farey fraction is isolated from the others.

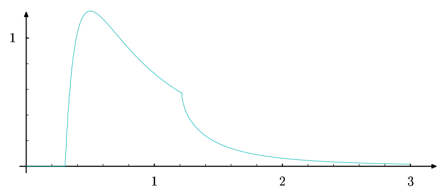

Figure 3.1: The density $g_1(t)$.

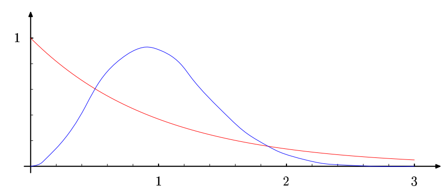

Figure 3.2: The Gaudin density compared to the poissonian density.

As noticed before, μ_h is the first term in the sequence of the h^{th} level of the d intervals probabilities $\{\mu_h^d\}_{d\geq 1}$ for the Farey sequence. In [CZ2002] it is proved that μ_h^d exists for any $d \geq 2$.

3.4. **The index of a Farey fraction and the support of μ_h.** We denote by $\mathcal{D}_h^d$ the support of μ_h^d, and in particular $\mathcal{D}_h = \mathcal{D}_h^1$ is the support of μ_h. It turns out that $\mathcal{D}_h^d$ has nice topological properties, unlike in most other cases of remarkable sequences for which the intervals distribution exists. In this section we look at $\mathcal{D}_h$ and in the next one at $\mathcal{D}_h^d$ for $d \geq 2$.

Let $\Phi_h\colon\ \mathcal{T} \to (0,\infty)^h$ be the map defined by

$$\Phi_h(x,y) = \tfrac{3}{\pi^2}\left(\tfrac{1}{L_0(x,y)L_1(x,y)}, \tfrac{1}{L_1(x,y)L_2(x,y)}, \dots, \tfrac{1}{L_{h-1}(x,y)L_h(x,y)}\right).$$

In [ABCZ2001] it is shown that $\mathcal{D}_h$ coincides with the closure of the range of the function $\Phi_h(x,y)$. From (3.3) one sees that $\mathcal{D}_1 = \big[3/\pi^2,\infty\big)$. For $h \geq 2$, $\mathcal{D}_h$ is strictly smaller than $\big[3/\pi^2,\infty\big)^h$. Taking into account the fact that $L_j(x,y)$ are defined recursively, we need to introduce an integer valued function that keeps the counting of the integer values involved. This is done by the map

$$\mathbf{k} : \mathcal{T} \to (\mathbb{N}^*)^h, \qquad \mathbf{k}(x,y) = \big(k_1(x,y),\dots,k_h(x,y)\big),$$

where, for $1 \le j \le h$,

$$k_j(x,y) = \left[\frac{1 + L_{j-1}(x,y)}{L_j(x,y)}\right].$$

The function $k_1(x,y)$ is used to define the index of a Farey fraction. If $\gamma = a/q < \gamma' = a'/q'$ are consecutive Farey fractions, then $\nu_Q(\gamma') := k_1\left(\frac{q}{Q}, \frac{q'}{Q}\right)$ is called *the index* of γ' (see Hall and Shiu [HS2001]). In Section 5.2 we present estimates for the moments of Farey fractions.

For any $\mathbf{k} \in (\mathbb{N}^*)^h$, we denote by

$$\mathcal{T}_{\mathbf{k}} = \left\{(x,y) \in \mathcal{T}\colon\ \mathbf{k}(x,y) = \mathbf{k}\right\}$$

the domains on which the map $\mathbf{k}(x,y)$ is locally constant. Another way to express $\mathcal{T}_{\mathbf{k}}$ is through the area-preserving transformation $T\colon\ \mathcal{T} \to \mathcal{T}$ defined by

$$T(x,y) = \left(y, \left[\tfrac{1+x}{y}\right]y - x\right).$$

One should notice that if $\gamma' < \gamma'' < \gamma'''$ are consecutive elements in $\mathcal{F}_Q$, then $T(\gamma', \gamma'') = (\gamma'', \gamma''')$. Then for any $\mathbf{k} = (k_1, \dots, k_h) \in (\mathbb{N}^*)^h$, we have

$$\mathcal{T}_{\mathbf{k}} = \mathcal{T}_{k_1} \cap T^{-1}\mathcal{T}_{k_2} \cap \cdots \cap T^{-h+1}\mathcal{T}_{k_h}.$$

(When $h = 1$ and $k \in \mathbb{N}^*$, we also write $\mathcal{T}_k = \left\{(x,y) \in \mathcal{T}\colon\ \left[\frac{1+x}{y}\right] = k\right\}$.) This shows that $\mathcal{T}_{\mathbf{k}}$ is a convex polygon and they form a partition of $\mathcal{T}$, that is $\mathcal{T} = \bigcup\limits_{\mathbf{k} \in (\mathbb{N}^*)^h} \mathcal{T}_{\mathbf{k}}$ and $\mathcal{T}_{\mathbf{k}} \cap \mathcal{T}_{\mathbf{k}'} = \emptyset$ whenever $\mathbf{k} \neq \mathbf{k}'$. We remark that when $h \ge 2$, some of the polygons $\mathcal{T}_{\mathbf{k}}$ are empty. More explicitly, for $h = 1$, we have

$$\mathcal{T}_k = \left\{(x,y) \in \mathcal{T}\colon\ \frac{1+x}{k+1} < y \le \frac{1+x}{k}\right\}$$

and for $h = 2$, if k and l are positive integers, then

$$\begin{aligned}\mathcal{T}_{k,l} =& \left\{(x,y) \in \mathcal{T}_k\colon\ \left[\frac{1+y}{ky - x}\right] = l\right\}\\ =& \left\{(x,y) \in \mathcal{T}_k\colon\ \frac{1+(l+1)x}{k(l+1)-1} < y \le \frac{1+lx}{kl-1}\right\}.\end{aligned}$$

Roughly speaking, $\mathcal{T}_k$ corresponds to the set of 3-tuples $(\gamma', \gamma'', \gamma''')$ of consecutive elements of $\mathfrak{F}_Q$ with the property that $\Delta(\gamma', \gamma''') = k$. Similarlly, $\mathcal{T}_{k,l}$ corresponds to the set of 4-tuples $(\gamma', \gamma'', \gamma''', \gamma^{iv})$ of consecutive elements of $\mathfrak{F}_Q$ with the property that $\Delta(\gamma', \gamma''') = k$ and $\Delta(\gamma'', \gamma^{iv}) = l$. We remark that $\mathcal{T}_{1,1} = \emptyset$, and also $\mathcal{T}_{k,l} = \emptyset$ whenever both k and l are ≥ 2 except in the cases $(k,l) \in \left\{(2,2); (2,3); (2,4); (3,2); (4,2)\right\}$. In Figures 3.3 and 3.4 one can see the polygons $\mathcal{T}_{\mathbf{k}}$ for $h = 1$ and $h = 2$.

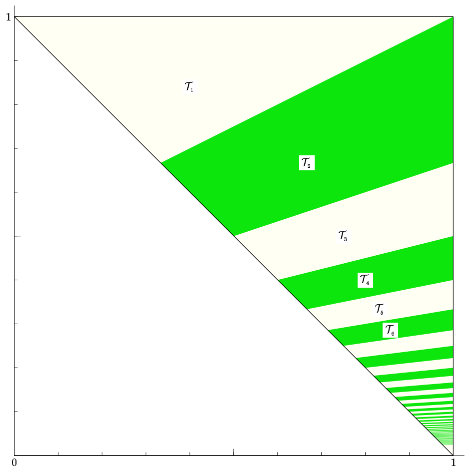

Figure 3.3: The polygons $\mathcal{T}_{\mathbf{k}}$ for $h = 1$.

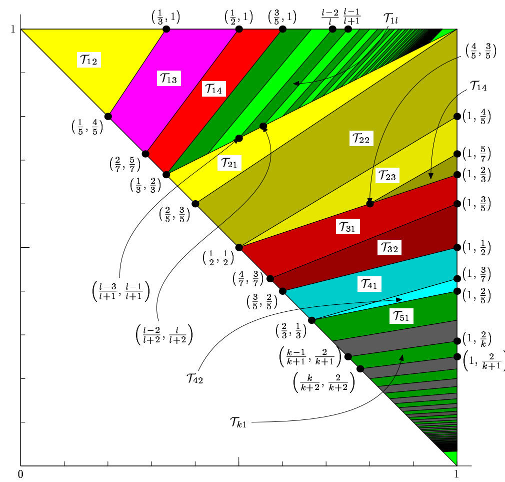

Figure 3.4: The polygons $\mathcal{T}_{\mathbf{k}}$ for $h = 2$.

The map $\Phi_2(x, y)$ transforms each $\mathcal{T}_k$ with $k \geq 2$ into a curved edge quadrangle and $\Phi_2(\mathcal{T}_1)$ is an unbounded curved edge triangle. Each of these sets is symmetric with respect to $u = v$ (u, v being the variables of the system of coordinates in which $\mathcal{D}_2$ is drawn) and their union is the support $\mathcal{D}_2$. The precise shape of D_2 is shown in Figure 3.7. It looks like a swallow with the top of the beak at $(\frac{3}{\pi^2}, \frac{3}{\pi^2})$ and a one-fold tail along the diagonal $u = v$. The lines $u = \frac{3}{\pi^2}$ and $v = \frac{3}{\pi^2}$ are asymptotes to the wings. The tail looks like a collection of diamonds parallel to each other, with two vertices symmetric with respect to $u = v$ and the other two vertices on the line $u = v$ arranged in such a way that the n-th and the $(n + 4)$-th diamonds have a common vertex. Formulae for the edges of all these constituents of $\mathcal{D}_2$ are given explicitly in [ABCZ2001]. We remark that each of these curves is an algebraic curve.

For $h \geq 3$ the support D_h looks more complicated, but we can look at the plotting of the projection of D_h on the 2-dimensional plane given by the first and last component. We denote this projection by $\tilde{D}_h$. In particular we have $\tilde{D}_2 = D_2$.

A picture that approximates D_3 with the points that come from $\mathcal{F}_Q$ with $Q = 300$ is shown in Figure 3.5. In passing from $h = 2$ to $h = 3$ the swallow seem to have suffered some kind of a metamorphosis losing its tail. Actually it is easy to see that the tail is

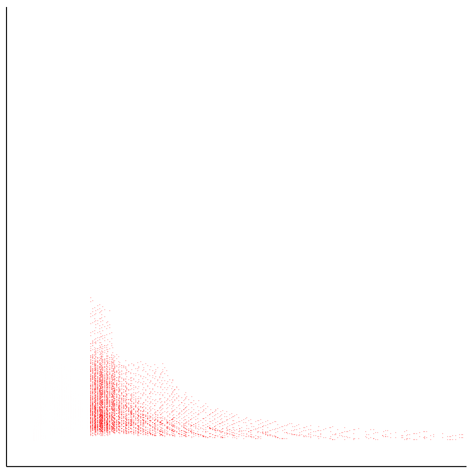

Figure 3.5: The projection of the support of μ_3 on the Oxz plane for $Q = 300$.

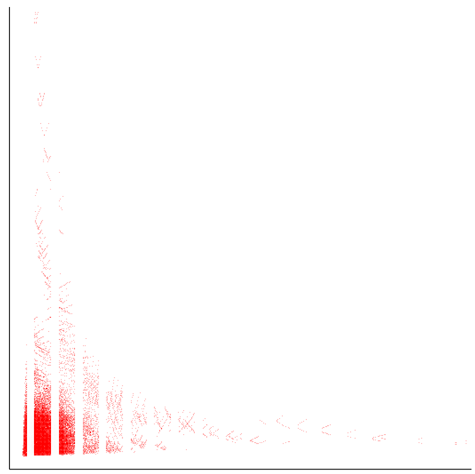

Figure 3.6: The projection of the support of μ_{11} on the Oxz plane for $Q = 300$.

lost for good, in the sense that no other $\tilde{D}_h$ will have a tail along the diagonal $u = v$. Indeed, a point with a large coordinate u comes from an $(h+1)$-tuple $(\frac{a_0}{q_0}, \frac{a_1}{q_1}, \dots, \frac{a_h}{q_h})$ of consecutive Farey fractions with $q_j \le Q$, for $0 \le j \le h$, with the property that $\frac{a_1}{q_1} - \frac{a_0}{q_0} = \frac{1}{q_0 q_1}$ is much larger than $\frac{1}{Q^2}$. Thus one of the denominators q_0 and q_1 will be much smaller than Q. Now the points with denominators much smaller than Q are far away from each other. So for any fixed h and $Q \to \infty$ we can not have one of q_0, q_1 small and also one of q_{h-1}, q_h small. Hence no $\tilde{D}_h$ with $h \ge 3$ will have a tail along the diagonal. For $h = 2$ the pairs (u, v) come from triplets $(\frac{a_0}{q_1}, \frac{a_1}{q_1}, \frac{a_2}{q_2})$ and here the middle fraction $\frac{a_1}{q_1}$ contributes to both coordinates u and v. So when q_1 is small we get a point close to the diagonal and this is how the tail of the swallow is obtained. We also remark that as h increases, the support of μ_h becomes more diffused. An example is presented in Figure 3.6. In the two pictures, 3.5 and 3.6, different scales were adapted to present the central parts of the projections. For guidance, one can use the fact that the beaks of "the animals" are the same.

3.5. **The support of μ_h^d.** For the intervals of a third distribution (the case $d = 2$) the analogue of $\Phi_2(x, y)$ is the map $\Phi_h^2\colon\ \mathcal{T} \to (0, \infty)^h$ defined by

$$\Phi_h^2(x, y) = \frac{3}{\pi^2}\left(\frac{k_1(x,y)}{L_0(x,y)L_2(x,y)}, \frac{k_2(x,y)}{L_1(x,y)L_3(x,y)}, \dots, \frac{k_h(x,y)}{L_{h-1}(x,y)L_{h+1}(x,y)}\right).$$

In particular, when $h = 2$, $\mathcal{D}_2^2$ is the image of $\Phi_2^2\colon\ \mathcal{T} \to \mathbb{R}^2$, $\Phi(x,y) = \frac{3}{\pi^2}\left(\frac{k}{xz}, \frac{l}{yt}\right)$, in which for any $(x,y) \in \mathcal{T}_{k,l}$, the variables z and t are given by $z = x - ky$, $t = y - lt$. A throughout computation allows to find explicitly the boundaries of $\Phi_2^2(\mathcal{T})$. The image obtained is shown in Figure 3.8. It is the two-fold tail swallow. All the equations of the boundaries of $\Phi_2^2(\mathcal{T}_{k,l})$ are either of the form $\frac{3}{\pi^2} \cdot \frac{et}{a+bt+c\sqrt{t(t-d)}}$, with t in a certain interval that can be unbounded, or the symmetric with respect to $x = y$ of such a curve. Here a, b, c, d, e are integers. The map $\Phi_2^2(x,y)$ has a "symmetrisation" property. This makes $\Phi_2^2(\mathcal{T}_{n,m})$ to be symmetric with to the first diagonal $u = v$ to $\Phi_2^2(\mathcal{T}_{m,n})$, for any $m, n \geq 1$. The "quadrangle" $\Phi_2^2(\mathcal{T}_{2,2})$ is the single nonempty domain $\Phi_2^2(\mathcal{T}_{k,l})$ that has $u = v$ as axis of symmetry. The top of the beak of the swallow $\mathcal{D}_2^2$ has coordinates $(6/\pi^2, 6/\pi^2)$. The asymptotes of the wings are $x = 6/\pi^2$ and $v = 6/\pi^2$.

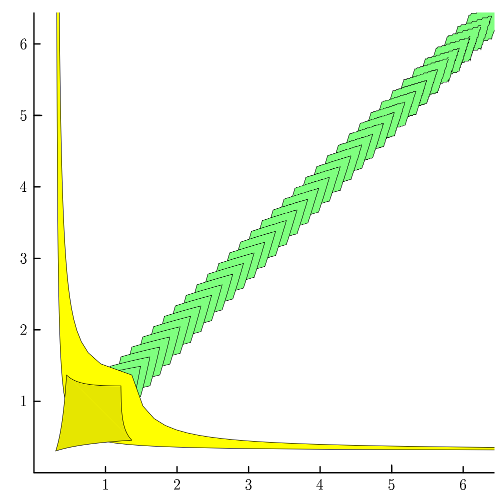

Figure 3.7: The support of μ_2^2.

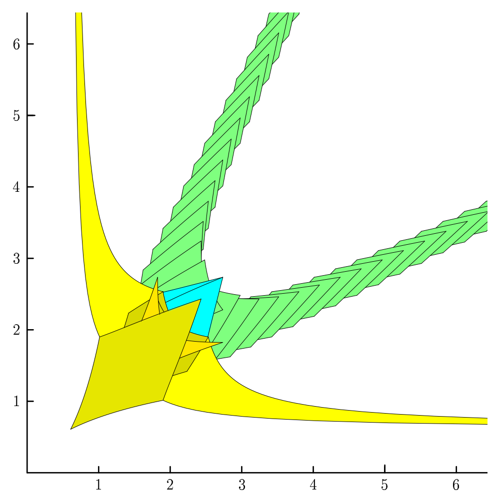

Figure 3.8: The support of μ_2^3.

For larger intervals, that is for $d \geq 3$, numerical computations show that the support $\mathcal{D}_2^d$ also looks like a swallow, which always has a three fold tail. As d increases, $\mathcal{D}_2^d$ departs more and more from the origin, with the coordinates of the beak in arithmetic progression situated always on the principal diagonal. Figures 3.9 and 3.10 present a picture of $\mathcal{D}_2^d$ for $d = 3$ and $d = 11$ obtained from the intervals of $\mathfrak{F}_Q$ with $Q = 300$. (In order to get a better understanding of the shapes, different scales were used in the two pictures.)

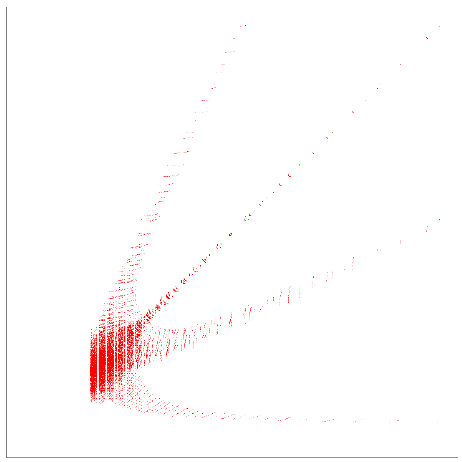

Figure 3.9: Pairs of neighbor intervals of a fourth.

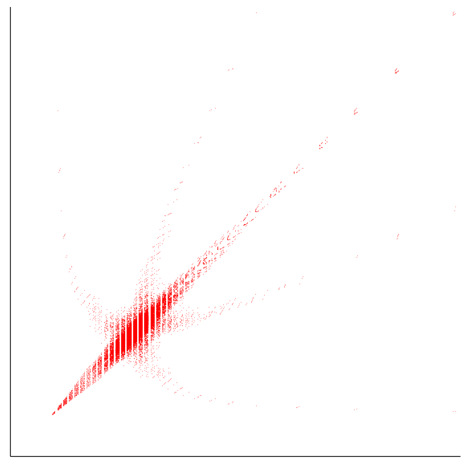

Figure 3.10: Pairs of neighbor intervals of an eleventh.

3.6. **A view of $\mathfrak{F}_Q$ from the outside.** A different approach to understand the distribution of $\mathfrak{F}_Q$ was taken by Kargaev and Zhigljavsky [KZ1997], [KZ1996] who studied the distance function from any $x \in (0,1)$ to $\mathfrak{F}_Q$. Putting $\rho_Q(x) := \min_{\gamma\in\mathfrak{F}_Q} |x-\gamma|$, among other things, they derived the asymptotic distribution of $Q^2\rho_Q(x)$. More precisely, as $Q \to \infty$, for any $\lambda > 0$,

$$\mu\big(\{x \in [0,1]:\ Q^2\rho_Q(x) \le \lambda\}\big) \to \int_0^\lambda \rho(t)\,dt\,,$$

where $\mu(\cdot)$ is the Lebesgue measure and the density $\rho(t)$ is given by:

$$\rho(t) = \begin{cases} \frac{6}{\pi^2}, & \text{if } 0 \le t \le \frac{1}{2} \\ \frac{6}{\pi^2 t}\big(1+\log(2t)-t\big), & \text{if } \frac{1}{2} \le t \le 2 \\ \frac{3}{\pi^2 t}\big(2\log(4t) - 4\log(\sqrt{t}+\sqrt{t-2}) - (\sqrt{t}-\sqrt{t-2})^2\big), & \text{if } 2 \le t \le \infty\,. \end{cases}$$

(There is an inadvertence in this formula in [KZ1997].) Notice that $\rho(t)$ is closer to the exponential density, having a large constant *attraction interval* near zero. This confirms the fact that Farey sequences are good approximants of irrational numbers.

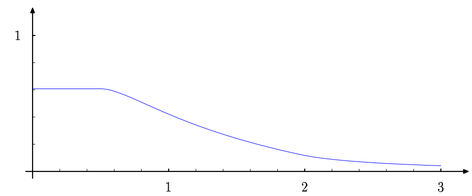

Figure 3.11: The density $\rho(t)$.

4. **Farey fractions with denominators in arithmetic progressions**

Here we present some results on the set of Farey fractions with denominators in an arithmetic progression. Let $c < d$ be nonnegative integers and denote

$$\mathfrak{F}_{Q,c,d} = \left\{ a/q \in \mathfrak{F}_Q : \ q \equiv c \pmod{d} \right\} .$$

As we saw in Section 2 that many statements on the distribution of $\mathfrak{F}_Q$ are equivalent to the Riemann hypothesis, one would naturally expect that $\mathfrak{F}_{Q,c,d}$ may be linked to the Generalized Riemann Hypothesis. A result in this direction was provided by Huxley [Hux1971]. We should remark that due to the symmetric role played by denominators and numerators in the basic relation $a''q' - a'q'' = 1$, which holds between any two consecutive fractions $\gamma', \gamma'' \in \mathfrak{F}_Q$, each statement on the distribution of the elements of $\mathfrak{F}_{Q,c,d}$ may be translated into an analogous one on the subset of Farey fractions whose numerators are in the arithmetic progression $c \pmod{d}$.

For $\mathfrak{F}_{Q,c,d}$ with $d \geq 2$ the fundamental relations (1.1) and (1.2) no longer hold true, though they may be replaced by other relations in a more complex form. This makes the study of $\mathfrak{F}_{Q,c,d}$ more involved, and a first step is to look at the subset of Farey fractions with odd denominators. We will say that a fraction is odd if its denominator is odd and write, more significantly, $\mathfrak{F}_{Q,\mathrm{odd}} = \mathfrak{F}_{Q,1,2}$.

4.1. **Distribution of Farey fractions with odd denominators.**

The fundamental relation $\Delta(\gamma', \gamma'') = a''q' - a'q'' = 1$, whenever $\gamma' = a'/q' < a''/q'' = \gamma''$ are consecutive elements of $\mathfrak{F}_Q$, fails when $\gamma' < \gamma''$ are consecutive in $\mathfrak{F}_{Q,\mathrm{odd}}$. This raises a natural question on how large is the number

$$N_{Q,\mathrm{odd}}(k) = \#\{(\gamma', \gamma'') : \ \gamma', \gamma'' \text{ consecutive in } \mathfrak{F}_{Q,\mathrm{odd}}\,,\ \Delta(\gamma, \gamma') = k\}\,,$$

for any integer $k \geq 1$. Knowing that

$$N_{Q,\mathrm{odd}} := \#\mathfrak{F}_{Q,\mathrm{odd}} = \frac{2Q^2}{\pi^2} + O(Q \log Q),$$

an answer to this problem was given by Haynes [Hay2001], who showed that for $k \in \mathbb{N}^*$, the following asymptotic frequency exists:

$$\rho_{\text{odd}}(k) := \lim_{Q\to\infty} \frac{N_{Q,\text{odd}}(k)}{\#\mathfrak{F}_{Q,\text{odd}}} = \frac{4}{k(k+1)(k+2)}\,. \tag{4.1}$$

Additionally, Haynes showed that the same frequency holds for the odd Farey fractions in a subinterval of $[0,1]$.

The estimate (4.1) can be written as

$$\rho_{\text{odd}}(k) = \begin{cases} 1/2 + \text{Area}(\mathcal{T}_1), & \text{if } k = 1,\\ \text{Area}(\mathcal{T}_k), & \text{if } k \geq 2, \end{cases}$$

revealing the subiacent geometry. Such a relation should be true, more generally, for tuples of consecutive odd fractions. The study of this structure is the object of [BCZ2002]. Thus, given the positive integers $\Delta_1, \dots, \Delta_h$, the problem requires to find the probability that an $(h+1)$-tuple of consecutive fractions $\gamma_j < \gamma_{j+1} < \cdots < \gamma_{j+h}$ in $\mathfrak{F}_{Q,\text{odd}}$ satisfies the conditions $\Delta(\gamma_j, \gamma_{j+1}) = \Delta_1, \dots, \Delta(\gamma_{j+h-1}, \gamma_{j+h}) = \Delta_h$. For this, one considers

$$N_{Q,\text{odd}}(\Delta_1, \dots, \Delta_h) = \#\left\{ j : \begin{array}{l} \gamma_j < \gamma_{j+1} < \cdots < \gamma_{j+h} \text{ consecutive in } \mathfrak{F}_{Q,\text{odd}} \\ \Delta(\gamma_{j+l-1}, \gamma_{j+l}) = \Delta_l,\ l = 1, \dots, h \end{array} \right\}$$

and sees whether for $h \geq 2$ the probability

$$\rho_{\text{odd}}(\Delta_1, \dots, \Delta_h) = \lim_{Q\to\infty} \frac{N_{Q,\text{odd}}(\Delta_1, \dots, \Delta_h)}{N_{Q,\text{odd}}}\,.$$

still exists. This is provided explicitly by

$$\rho_{Q,\text{odd}}(\Delta_1, \dots, \Delta_h) = \sum_{w \in \mathfrak{S}(\Delta_1,\dots,\Delta_h)} \text{Area}(\mathcal{T}_{k_1,\dots,k_{|w|-1}}) + O_h\left(\frac{\log^2 Q}{Q}\right), \tag{4.2}$$

where the index of summations runs over selected paths in a certain odd Farey tree with label branches satisfying a natural parity condition.

For $h = 2$, there are four cases, depending on the size of Δ_1, Δ_2.

1. If $\Delta_1 = 1$ and $\Delta_2 = 1$, then

$$\begin{aligned}\rho_{\text{odd}}(1,1) = {} & \sum_{k_1 \text{ even}} \text{Area}(\mathcal{T}_{k_1}) + \sum_{k_1 \text{ odd}} \text{Area}(\mathcal{T}_{k_1} \cap T^{-1}\mathcal{T}_1) + \sum_{k_2 \text{ odd}} \text{Area}(\mathcal{T}_1 \cap T^{-1}\mathcal{T}_{k_2}) \\ & + \sum_{k_2 \text{ even}} \text{Area}(\mathcal{T}_1 \cap T^{-1}\mathcal{T}_{k_2} \cap T^{-2}\mathcal{T}_1)\end{aligned}$$

2. If $\Delta_1 = 1$ and $\Delta_2 \geq 2$, then

$$\rho_{\text{odd}}(1, \Delta_2) = \sum_{k_1 \text{ odd}} \text{Area}(\mathcal{T}_{k_1} \cap T^{-1}\mathcal{T}_{\Delta_2}) + \sum_{k_2 \text{ even}} \text{Area}(\mathcal{T}_1 \cap T^{-1}\mathcal{T}_{k_2} \cap T^{-2}\mathcal{T}_{\Delta_2}).$$

3. If $\Delta_1 \geq 2$ and $\Delta_2 = 1$, then

$$\rho_{\text{odd}}(\Delta_1, 1) = \sum_{k_2 \text{ odd}} \text{Area}(\mathcal{T}_{\Delta_1} \cap T^{-1}\mathcal{T}_{k_2}) + \sum_{k_2 \text{ even}} \text{Area}(\mathcal{T}_{\Delta_1} \cap T^{-1}\mathcal{T}_{k_2} \cap T^{-2}\mathcal{T}_1).$$

4. If $\min(\Delta_1, \Delta_2) \geq 2$, then

$$\rho_{\text{odd}}(\Delta_1, \Delta_2) = \sum_{k_2 \text{ even}} \text{Area}(\mathcal{T}_{\Delta_1} \cap T^{-1}\mathcal{T}_{k_2} \cap T^{-2}\mathcal{T}_{\Delta_2}).$$

We remark that since neighbor Farey fractions are closely related, for random $\Delta_1, \dots, \Delta_h$ one should expect that $\rho_{\text{odd}}(\Delta_1, \dots, \Delta_h) = 0$ and even that $N_{Q,\text{odd}}(\Delta_1, \dots, \Delta_h) = 0$. This is certainly true in the case $h = 2$, in which $\rho_{\text{odd}}(\Delta_1, \Delta_2)$ are the entries of the following matrix:

$$\begin{pmatrix} \frac{91}{210} & \frac{17}{210} & \frac{11}{210} & \frac{4}{4\cdot5\cdot6} & \frac{4}{5\cdot6\cdot7} & \frac{4}{6\cdot7\cdot8} & \cdots \\ \frac{17}{210} & \frac{15}{210} & \frac{3}{210} & 0 & 0 & 0 & \cdots \\ \frac{11}{210} & \frac{3}{210} & 0 & 0 & 0 & 0 & \cdots \\ \frac{4}{4\cdot5\cdot6} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{4}{5\cdot6\cdot7} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{4}{6\cdot7\cdot8} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Using the Kloosterman machinery, the estimate (4.2) is extended in [BCZ2002] to the set of odd Farey fractions in a subinterval of $[0, 1]$. As was expected, the probability has the same main term, but the error term pays the price, being replaced by $O(Q^{-1/2+\epsilon})$.

4.2. **The relative size of consecutive denominators of Farey fractions in arithmetic progressions.** For any two consecutive Farey fractions $a'/q' < a''/q''$ one has $q' + q'' > Q$, but this is no longer true for consecutive elements of $\mathfrak{F}_{Q,\text{odd}}$. This raises the natural question to find the location of the points $(q'/Q, q''/Q)$ where q' and q'' are consecutive denominators of fractions from $\mathfrak{F}_{Q,\text{odd}}$. For any nonnegative integers $c < d$, we consider the set

$$\mathcal{C}_{Q,c,d} = \{(q'/Q, q''/Q) \colon\ q', q'' \text{ consecutive denominators of fractions in } \mathfrak{F}_{Q,c,d}\}\,.$$

In particular, we write $\mathcal{O}_Q = \mathcal{C}_{Q,1,2}$ and $\mathcal{E}_Q = \mathcal{C}_{Q,0,2}$. The authors jointly with Iordache [CIZ2003] show that as $Q \to \infty$, the limit of the sets $\mathcal{O}_Q$ is dense in the region bounded by the lines $y = 1$, $x = 1, 2x + y = 1$, $2y + x = 1$.

A picture of the sets $\mathcal{O}_Q$ is shown in Figure 4.1. One can show that $\mathcal{O}$, the closure of the limit set of $\mathcal{O}_Q$'s as $Q \to \infty$, is the union

$$\mathcal{O} = \mathcal{T} \cup \bigcup_{k=1}^{\infty} \mathcal{V}_k \,,$$

where $\mathcal{V}_1$ is the triangle with vertices $(0,1)$; $\left(\frac{1}{3}, \frac{1}{3}\right)$; $(1,0)$, and for $k \geq 2$ the set $\mathcal{V}_k$ is the quadrilateral with vertices $\left(\frac{k-1}{k+1}, 1\right)$; $\left(\frac{k}{k+2}, \frac{k}{k+2}\right)$; $\left(1, \frac{k-1}{k+1}\right)$; $(1,1)$. The quadrilateral $\mathcal{V}_k$ lies over $\mathcal{V}_{k-1}$ for $k \geq 3$. Numerical calculations performed with relatively small values already show their shadow over $\mathcal{O}$.

A more complex analysis is needed for the similar problem on the even Farey fractions. This is due to the fact that between two consecutive even fractions from $\mathfrak{F}_Q$ there may exist many odd fractions. Indeed, $1/2$ has in $\mathfrak{F}_Q$ as many as $\left[\frac{Q}{4}\right] + a$ odd neighbors on each side, where $a = 0, 1, 1, 2$ for $Q \equiv 0, 1, 2, 3 \pmod 4$, respectively. Though, in [CZ2003] the authors prove that $\mathcal{E}$, the closure of the limit as $Q \to \infty$ of the sets $\mathcal{E}_Q$, is the same quadrangle with vertices $(1,1)$; $(0,1)$; $(1/3, 1/3)$; $(1,0)$, exactly as in the odd case.

For $d \geq 3$, numerical calculations seem to suggest that $\mathcal{C}_{Q,c,d}$, for any $c < d$, is a larger and larger quadrangle that tends to cover the unit square as d increases. But only the first part of this statement seems to be true, since $\mathcal{C}_{3,12}$, the limit as $Q \to \infty$ of the sets $\mathcal{C}_{Q,3,12}$, appears to be a hexagon with vertices $(1,1)$; $(0,1)$; (u,v); $(1/5, 1/5)$; (v,u); $(1,0)$, where u, v are rational numbers, $u \approx 7/100$ and $v \approx 19/50$.

We return now to the set of odd Farey fractions. In order to take into account the contribution of different pairs (q', q'') of consecutive denominators of fractions in $\mathfrak{F}_{Q,\text{odd}}$, we remark that they are either inherited from pairs of odd fractions consecutive in $\mathfrak{F}_Q$, or in $\mathfrak{F}_Q$ there exists exactly one even fraction in between the fractions with denominators q', q''. Some, but not all, of this last type of pairs still satisfy the condition $q' + q'' > Q$, as all pairs of first type do. It turns out that a pair (q', q'') of consecutive denominators of fractions in $\mathfrak{F}_{Q,\text{odd}}$ satisfies the condition $q' + q'' > Q$ with probability $5/6$.

In [CIZ2003] it is also calculated the limit density of the points $(q'/Q, q''/Q)$, where q' and q'' are consecutive denominators of fractions from $\mathfrak{F}_{Q,\text{odd}}$ in the unit square, as