SOME REMARKS CONCERNING THE COVERING NUMBERS OF CLOSED MANIFOLDS

by
Mihaela Todea

Abstract: In this paper we shall present some notions reffering to the covering numbers of closed manifolds and we shall make a classification of 4-dimensional closed manifolds, M^4, with $b_1(M) = 4$. We shall also demonstrate some results of n-dimesional closed manifolds, with P_1 – covering number equal to 4.

For M a closed, connected n-manifold, there are some numerical invariants, i.e. the notion of category and that of covering number for M. These invariants are strictly connected to the topologycal structure of M. Thus, we shall complete the classical results of 3-topologycal manifolds and then we shall demonstrate some theorems of characterization in superior dimensions.

1. In the beginning we shall introduce to you some notion often used within this article.

We shall suppose that manifolds are compact and connected. Being given an n-dimensional manifold, M^n, ∂M and Int(M) represents the boundary and the interior of M. M is said to be closed if ∂M is empty. If P is a compact polyhedron in M, then N(P,M) represents a regular manifold of p in M. The interior and closing of P in M will be denoted by Int(P,M) and cl(P,M).

An n-ball B^n is an n-dimensional manifold at an n-simplex standard. The n-dimensional sphere S^n is the border of an n+1 ball. For $n \geq 2$, the symbol $S^1 \otimes S^{n-1}$(respectively $S^1 \otimes B^{n-1}$), means either the topologycal product $S^1 \otimes S^{n-1}$(respectively $S^1 \otimes B^{n-1}$), or the connected fibre S^{n-1}(resp. B^{n-1}) over the sphere S^1. Even more, $k(S^1 \otimes S^{n-1})$ şi $k_\partial(S^1 \otimes B^{n-1})$ represents the connected sum and the border of the connected sum of k products $S^1 \otimes S^{n-1}$ şi $S^1 \otimes B^{n-1}$ (respectively). If k=0, we choose $(S^1 \otimes S^{n-1})= S^n$ (n-sphere) şi $k_\partial(S^1 \otimes B^{n-1})= B^n$ (n-ball).

An n-dimensional closed manifold M is said to have a topologycal handle if M is homeomorphic to a connected sum M'# $(S^1 \otimes S^{n-1})$ for M′ a closed manifold. Contrary M is said to be an n-manifold without a handle . We say that M trivial (resp. non trivial) if M is (resp. is not) an n-sphere. A closed n-manifold is prime if and only if $M=M_1\#M_2$ implies M_1 or M_2 trivial. M is irreducible if and only if each (n-1)- sphere in M edges an n-ball in M. Obviously the n-dimensional irreducible manifolds are prime and the prime manifolds without a handle are irreducible.

2. We consider X a topological space. The Lusternik-Schnirelmann category of X, denoted by cat(X) is the smallest number of closed (resp. open) contractible subsets which covers X. The category is an invariant of the omotopy type, i.e. if X and Y are two topological spaces with the same type of omotopy, then cat(X)=cat(Y). Recall that the closed n-manifold has the 2 category if and only if is an n-sphere of omotopy..

Consider M^n a closed n-dimensional manifold. Another classical invariant of M is the minimal number C(M) of the necessary n- balls to cover M. We have the inequalities:

$$2 \leq cat(M) \leq C(M) \leq n+1$$

Even more C(M) coincides with cat (M) whenever cat (M)$\geq \frac{1}{2}n+2$ şi $n \geq 4$.

The main generalities about these invariants are given their closed connections to the classical problems of the topological manifolds like in Poincaré and Schoenfliës. The minimal covering ball of the manifolds has also been researched by many authors before the ones mentioned above. In the present section we consider the coverage ball of the closed n-manifolds of which n-balls have features of the beautiful crossing point in the sense of the following definitions:

Definition 1

Be it M^n a closed n-manifold and be it $B=\{B_i/i\in I\}$ a finite set of balls within M so that $\cup B=M$. Then B is called a P_0 –covering ball (resp. P_1-covering ball) of M if and only if $B_i \cap B_j=\partial B_i \cap \partial B_j$ has (n-1)- manifolds (respectively (n-1)- balls) as connected components for each $i,j\in I$, $i\neq j$.

The concept of P_0 and P_1- coverings ball of M have firstly been introduced by K. Kobayashi and Y . Tsukui, "The ball coverings of manifolds " J. Maths. Soc. Japan 28 (1976), 133-143 and also by M. Ferri and C. Gagliardi, "Strong ball coverings of manifolds", Atti. Semin. Mat. Fis. Univ. Modena 28 (1979), 289-293.

Definition 2

With the above notations, B is called a P_2- covering ball of M^n if and only if the components of the intersection $\cap\{B_j|j\in J\}=\cap\{\partial B_j|j\in J\}$are (n-h+1)- balls for $\forall$ subset J of I with card (J)=h$\in N^0_{n+1}-\{1\}$.

The notion of P_2- coverins ball of a manifold has been firstly mentioned and introduced by M. Pezzana, "Sulla struttura topologica delle varietá compatte", Atti Sem. Mat. Fis. Univ. Modena 23 (1974) 269-277.

We define now P_r.- the covering number, written by $b_r(M)$, of $M(r\in N_2)$, as the minimum of the n-balls number of P_r.- the covering ball of M, i.e.:

$b_r(M)=\min\{card(B)|B$ is a P_r covering ball of $M\}$

Obviously $2 \le cat(M) \le C(M) \le b_0(M) \le b_1(M) \le b_2(M)$ and $b_0(M) = 2$ (respectively $b_1(M)=3$, $\dim(M) = n \ge 2$) if and only if M^n is trivial. The following results are well-known:

Proposition 1

a) If M^n- is a n-dimensional closed manifold, then $b_2(M)=n+1$ (conform[P]);

b) If M^n-is a n- sphere of omology and $b_0(M) \le 3$, then M^n is trivial ([KT], prop.3.2.);

If M^n ($n\ge5$) is a contractible n-dimensional manifold, then $b_0(M) \le 3$ ([KT]);

c) If M^n ($n=p+q$, $p,q\ge1$) is a p-sphere connected over a q- sphere then $b_0(M) \le 4$. More, if M has a transversal cut then $b_0(M)\le3$ ([T1]).;

d) If M^4 is a 4-omotopy sphere then $b_0(M\#k(S^2XS^2)) \le 3$ for $k\ge0$, $k\in N$ ([KT], prop.3.7.);

e) If M^3 is a closed 3-dimensional manifold, then $b_0(M) = 3$ if and only if M is homeomorphic to $k(S^1\otimes S^2)$ for $k\ge1$, $k\in N$ ([KT], th4.3.; [HM]-main theorem).

Be now B a P_r – covering ball of a closed n-manifold $M(r\in N_1)$. Then the intersection of whichever h ($3\le h\le n+1$) n- balls of B, has (n-h+1) manifolds as connected components. The demonstration can be obtained directly using the same methods (transversality and general position) described by W. Singhof, "Minimal coverings of manifolds with balls" Manuscr. Math. 29 (1979), 385-415, for the categorical completions of the manifolds :

For a 1 – closed manifold M^1 ($\approx S^1$), the above mentioned definitions coincide, so that $cat(M)=C(M)=b_0(M)=b_1(M)=b_2(M)=2$. In 2-dimensional, each P_1- covering ball of a closed surface is only a P_2- covering ball. Thus we have: $cat(M) = C(M) = b_0(M) = b_1(M) = b_2(M) = 3$ if and only if M is not trivial and $cat(S^2)=C(S^2)=b_0(S^2)=2$, $b_1(S^2)=b_2(S^2)=3$.

For M^3 a closed 3-dimensional manifold, P_r – covering numbers of M are calculated as follows:

1) $b_0(M) = 2$ (respectively $b_1(M) = 3$) if and only if M is trivial;

2) $b_0(M) = 3$ if and only if $M\approx k(S^1\otimes S^2)$ for $k\ge1$, $k\in N$;

3) If M is not trivial, 3-dimensional manifold without handle, then $b_0(M) = b_1(M) = b_2(M) = 4$.

Proposition 2

Let be M^n an n-orientable closed manifold ($n \ge 4$) with $b_1(M)=4$. If $B=\{B_i|i\in N_3\}$is a P_1- covering ball of M, then $H_q(M) = 0$ ($2 \le q \le n-2$) and $H_1(M)\approx H_{n-1}(M)\approx(r-1)Z$ where r is the number of the components of $B_i \cap B_j$ for each $i,j\in N_3$ ($i\ne j$).

Proof

Let be (i, j, h, k) a permutation of N_3. The reunion $B_i \cup B_j$ is divided in the suspension $\sum(B_i \cap B_j)$ of the intersection $B_i \cap B_j$. Since $B_i \cap B_j$ is a disjointed reunion of (n-1) - balls, $\sum(B_i \cap B_j)$ is divided within the graph G formed from 2 point united by r edges, where r is the number of the components of $B_i \cap B_j$. This implies the facts that:

$B_i \cup B_j \approx (r-1)_\partial(S^1 X B^{n-1})$ when $B_i \cup B_j$ is a regular neighborhood of G in n – orientable manifold M.

Noticing that $\partial(B_i \cup B_j) = \partial(B_h \cup B_k)$, we have also $B_h \cup B_k \approx (r-1)_\partial(S^1 X B^{n-1})$. The row (M, V) of the pair $(B_i \cup B_j, B_h \cup B_k)$ gives us $H_q(M) = 0$ $(3 \le q \le n-2)$ if $n > 4$ and $H_{n-1}(M) \approx (r-1) Z$ if $n \ge 4$. Even more, using the exact row:

$0 = H_2(B_i \cup B_j) \oplus H_2(B_h \cup B_k) \to H_2(M) \to H_1((r-1)(S^1 X S^{n-2})) \to H_1(B_i \cup B_j) \oplus H_1(B_h \cup B_k) \to H_1(M) \to 0$, we obtain:

$0 \to H_2(M) \to (r-1)Z \to 2(r-1)Z \to H_1(M) \to 0$, from where $H_2(M)$ doesn't have torsion. According to the duality of Poincaré, we have:

$H_1(M) \approx H^{n-1}(M) \approx TH_{n-1}(M) \oplus TH_{n-2}(M) \approx (r-1)Z$, when $H_{n-1}(M) \approx (r-1)Z$, $H_{n-2}(M) = 0$ for $n > 4$ and $H_2(M)$ doesn't have torsion for n = 4. Thus, the exact row:

$0 \to H_2(M) \to (r-1)Z \to 2(r-1)Z \to (r-1)Z \to 0$ gives us $H_2(M)=0$. This completes the demonstration

Proposition 3

a) Let be M^n an n-dimensional closed simply connected manifold. If $b_1(M) \le 4$ Then M is trivial.

b) Let be M^{2n} a 2n –dimensional closed, orientable manifold with Euler's characteristic $\chi(M^{2n}) = 2$. If $b_1(M) \le 4$ Then M is trivial.

Proof

a) For such an M manifold, the row of isomorphic groups:

$\Pi(M) \approx H_1(M) \approx (r-1)Z \approx 0$ gives us r = 1.

Thus, $B_i \cup B_j \approx B_h \cup B_k$ is an n – ball and M is the reunion of two n–balls with a common boundary .

c) We have:

$\chi(M^{2n}) = 1 - (r-1) + (-1)^{2n-1}(r-1) + (-1)^{2n} = 4 - 2r = 2$, then r=1

Proposition 4

a) Let be M^4 a closed manifold 4-dimensional. Then $b_1(M)=4$ if and only if M is homeomorphic at $k(S^1 \otimes S^3)$ for $k \ge 1$, $k \in N$.

b) If M^4 is a closed manifold 4-dimensional, non trivial, without a handle then $b_1(M) = 5$.

Proof

(a)Let be $B = \{B_i | i \in N_3\}$ a P_1 – covering ball of M^4. Observing that $B_0 \cup B_1 \approx B_2 \cup B_3 \approx (r\text{-}1)_\partial(S^1 \otimes B^3)$, the request immediately results from the theorem 2 from [Mo];

(b)Is a direct consequence of (a).

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Author:

Mihaela Todea-„1 Decembrie 1918" University of Alba Iulia, Romania, Departament of Mathematics, e-mail: mtodea@uab.ro