Acta Universitatis Apulensis No. 35/2013
ISSN: 1582-5329 pp. 7-16

ON GROWTH PROPERTIES OF ITERATED ENTIRE FUNCTIONS

Arkojyoti Biswas

Abstract. In this paper we study growth properties of iterated entire functions which improves some earlier results.

2000 *Mathematics Subject Classification*: 30D35.

Keywords: Entire Functions, iteration, growth.

1. Introduction, definitions and notations

Let f and g be non constant entire functions defined in the open complex plane $\mathbb{C}$ and $M(r,f)=\max\{|f(z)|:|z|=r\}$.The order and lower order of f are defined in the following way.

Definition 1. *The order ρ_f and lower order λ_f of an entire function f are defined as follows:*

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} \quad \textit{and} \quad \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}$$

where we use the following notation (cf. [7]):

$$\log^{[k]}x=\log(\log^{[k-1]}x) \textit{ for } k=1,2,3\ \ldots \textit{ and } \log^{[0]}x=x.$$

Definition 2. *The type σ_f of an entire function f is defined as:*

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\rho_f}},0<\rho_f<\infty.$$

In line of Lahiri and Banerjee [5] we define the iteration of $f(z)$ with respect to $g(z)$ as follows:

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z))=f(g_1(z))\\
f_3(z) &= f(g(f(z)))=f(g_2(z))=f(g(f_1(z)))
\end{aligned}$$

$$\begin{aligned} & \ldots && \ldots && \ldots \\ f_n(z) &= f(g(f\ldots\ldots(f(z) \text{ or } g(z))\ldots\ldots)) \\ & \quad \text{according as } n \text{ is odd or even,} \end{aligned}$$

and so

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(f(z)) = g(f_1(z)) \\ g_3(z) &= g(f_2(z)) = g(f(g(z))) \\ \ldots & \quad \ldots \quad \ldots \\ g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))). \end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

It is well known that

$$\lim_{r\to\infty} \frac{M(r, fog)}{M(r, f)} = \lim_{r\to\infty} \frac{M(r, fog)}{M(r, g)} = \infty.$$

Clunie [1] discussed on the behaviour of

$$\frac{\log M(r, fog)}{\log M(r, f)} \quad \text{and} \quad \frac{\log M(r, fog)}{\log M(r, g)} \ as\ r \to \infty.$$

Song and Yang [10] worked on

$$\frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, f)} \quad \text{and} \quad \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(r, g)} \ as\ r \to \infty.$$

Replacing maximum modulus functions by Nevanlinna's characteristic functions Clunie [1] proved for any two transcendental entire functions defined in the open complex plane $\mathbb{C}$,

$$\lim_{r\to\infty} \frac{T(r, fog)}{T(r, f)} = \infty \quad \text{and} \quad \lim_{r\to\infty} \frac{T(r, fog)}{T(r, g)} = \infty.$$

Singh [8] proved some comparative growth properties of $\log T(r, fog)$ and $T(r, f)$. Singh [8] also raised the problem of investigating the comparative growth of $\log T(r, fog)$ and $T(r, g)$ and some results on the comparative growth of $\log T(r, fog)$ and $T(r, g)$ are proved in Lahiri [3].

Since $M(r,f)$ and $M(r,g)$ are increasing functions of r, Singh and Baloria [9] asked whether for any two entire functions f, g and for sufficiently large $R = R(r)$,

$$\limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, f)} < \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log^{[2]} M(R, g)} < \infty.$$

Singh and Baloria [9], Lahiri and Sharma [4], Liao and Yang [6] worked on this question.

In this paper we study growth properties of iterated entire functions which improves some results of Liao and Yang [6].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *Let $f(z)$ and $g(z)$ be two entire functions with non zero finite orders ρ_f and ρ_g respectively. Then for any $\varepsilon > 0$ and for all sufficiently large values of r*

$$\log^{[n]} M(r, f_n) \leq \begin{cases} (\rho_f + \varepsilon) \log M(r, g) + O(1) & \text{when } n \text{ is even} \\ (\rho_g + \varepsilon) \log M(r, f) + O(1) & \text{when } n \text{ is odd.} \end{cases}$$

Lemma 1 follows from Lemma 2.4 of Dutta [2] on putting $p = 1$.

Lemma 2. *Let $f(z)$ and $g(z)$ be two entire functions with non zero finite lower orders λ_f and λ_g respectively. Then for any $0 < \varepsilon < \min\{\rho_f, \rho_g\}$ and for all sufficiently large values of r*

$$\log^{[n]} M(r, f_n) \geq \begin{cases} (\lambda_f - \varepsilon) \log M\left(\frac{r}{2^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g - \varepsilon) \log M\left(\frac{r}{2^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd.} \end{cases}$$

Lemma 2 follows from Lemma 2.5 of Dutta [2] on putting $p = 1$.

3. Theorems

In this section we present the main results of our paper.

Theorem 3. *Let $f(z)$ and $g(z)$ be two entire functions such that* $0<\lambda_f\leq\rho_f<\infty$ *and* $\lambda_g\leq\rho_g<\infty$ *. Then for each* $\alpha\in[0,\infty)$ *and* $0<p<\min\{(1+\alpha)\rho_g,(1+\alpha)\rho_f\}$

$$(i)\limsup_{r\to\infty}\frac{\left\{\log^{[n]}M(r,f_n)\right\}^{1+\alpha}}{\log^{[2]}M(\exp(r^p),f)} = \infty$$

$$(ii)\limsup_{r\to\infty}\frac{\left\{\log^{[n]}M(r,f_n)\right\}^{1+\alpha}}{\log^{[2]}M(\exp(r^p),g)} = \infty.$$

Proof. (i) Case 1: When n is even.

From Lemma 2 we get for a sequence of values of r tending to infinity that

$$\log^{[n]}M(r,f_n)\geq(\lambda_f-\varepsilon)\log M\left(\frac{r}{2^{n-1}},g\right)+O(1)$$

$$>(\lambda_f-\varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_g-\varepsilon}+O(1). \tag{1}$$

Now from the definition of order we obtain for all sufficiently large values of r that

$$\log^{[2]}M(\exp(r^p),f)\leq(\rho_f+\varepsilon)\log\exp(r^p)$$

$$i.e., \log^{[2]}M(\exp(r^p),f)\leq(\rho_f+\varepsilon)\ r^p. \tag{2}$$

From (1) and (2) we obtain for a sequence of values of r tending to infinity that

$$\frac{\left\{\log^{[n]}M(r,f_n)\right\}^{1+\alpha}}{\log^{[2]}M(\exp(r^p),f)}>\frac{(\lambda_f-\varepsilon)^{(1+\alpha)}\left(\frac{r}{2^{n-1}}\right)^{(\rho_g-\varepsilon)(1+\alpha)}+O(1)}{(\rho_f+\varepsilon)\ r^p}. \tag{3}$$

As $0<p<(1+\alpha)\rho_g$ and $\varepsilon>0$ is arbitrary, for each $\alpha\in[0,\infty)$ we can choose $\varepsilon>0$ such that $p<(\rho_g-\varepsilon)(1+\alpha)$ and therefore from (3) we obtain

$$\limsup_{r\to\infty}\frac{\left\{\log^{[n]}M(r,f_n)\right\}^{1+\alpha}}{\log^{[2]}M(\exp(r^p),f)}=\infty.$$

Case 2: When n is odd.

From Lemma 2 we get for a sequence of values of r tending to infinity that

$$\log^{[n]}M(r,f_n)\geq(\lambda_g-\varepsilon)\log M\left(\frac{r}{2^{n-1}},f\right)+O(1)$$

$$> (\lambda_g - \varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_f - \varepsilon} + O(1). \tag{4}$$

From (4) and (2) we obtain for a sequence of values of r tending to infinity that

$$\frac{\left\{\log^{[n]} M(r, f_n)\right\}^{1+\alpha}}{\log^{[2]} M(\exp(r^p), f)} > \frac{(\lambda_g - \varepsilon)^{(1+\alpha)}\left(\frac{r}{2^{n-1}}\right)^{(\rho_f - \varepsilon)(1+\alpha)} + O(1)}{(\rho_f + \varepsilon)\ r^p}. \tag{5}$$

As $0 < p < (1+\alpha)\rho_f$ and $\varepsilon > 0$ is arbitrary, for each $\alpha \in [0, \infty)$ we can choose $\varepsilon > 0$ such that $p < (\rho_f - \varepsilon)(1+\alpha)$ and therefore from (5) we obtain

$$\limsup_{r\to\infty} \frac{\left\{\log^{[n]} M(r, f_n)\right\}^{1+\alpha}}{\log^{[2]} M(\exp(r^p), f)} = \infty.$$

Hence in both the cases we get

$$\limsup_{r\to\infty} \frac{\left\{\log^{[n]} M(r, f_n)\right\}^{1+\alpha}}{\log^{[2]} M(\exp(r^p), f)} = \infty.$$

(ii) replacing f by g in (2) we get

$$\limsup_{r\to\infty} \frac{\left\{\log^{[n]} M(r, f_n)\right\}^{1+\alpha}}{\log^{[2]} M(\exp(r^p), g)} = \infty.$$

Remark 1. *Theorem 3 improves Theorem 4 of Liao and Yang [6].*

Remark 2. *If we take* $0 < p < \min\{(1+\alpha)\lambda_g, (1+\alpha)\lambda_f\}$ *instead of* $0 < p < \min\{(1+\alpha)\rho_g, (1+\alpha)\rho_f\}$ *then Theorem 3 remains valid as we see in the following theorem.*

Theorem 4. *Let* $f(z)$ *and* $g(z)$ *be two entire functions such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $\lambda_g \leq \rho_g < \infty$ *. Then for each* $\alpha \in [0, \infty)$ *and* $0 < p < \min\{(1+\alpha)\lambda_g, (1+\alpha)\lambda_f\}$

$$(i)\ \lim_{r\to\infty} \frac{\left\{\log^{[n]} M(r, f_n)\right\}^{1+\alpha}}{\log^{[2]} M(\exp(r^p), f)} = \infty$$

$$(ii)\ \lim_{r\to\infty} \frac{\left\{\log^{[n]} M(r, f_n)\right\}^{1+\alpha}}{\log^{[2]} M(\exp(r^p), g)} = \infty.$$

Proof of Theorem 4 is similar to Theorem 3 and so is omitted.

Theorem 5. *Let $f(z)$ and $g(z)$ be two entire functions of finite order such that $0 < \lambda_f$. If $\sigma_g < \infty$ then*

$$(i)\ \limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} < \infty \text{ if } n \text{ is even.}$$

(ii) Further if $\rho_g > \rho_f$, $\sigma_f < \infty$ and n is odd then

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} = 0.$$

Proof. (i) As n is even from Lemma 1 we get for all sufficiently large values of r that

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} \le \limsup_{r\to\infty} \frac{(\rho_f + \varepsilon)\log M(r, g) + O(1)}{\log^{[2]} M(\exp(r^{\rho_g}), f)}$$

$$\le \limsup_{r\to\infty} \frac{(\rho_f + \varepsilon)\log M(r, g) + O(1)}{r^{\rho_g}} \limsup_{r\to\infty} \frac{r^{\rho_g}}{\log^{[2]} M(\exp(r^{\rho_g}), f)}$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} \le (\rho_f + \varepsilon)\ \sigma_g\ \frac{1}{\lambda_f} \tag{6}$$

As $\varepsilon > 0$ is arbitrary from (6) we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} \le \rho_f\ \sigma_g\ \frac{1}{\lambda_f} < \infty.$$

(ii) As n is odd from Lemma 1 we get for all sufficiently large values of r that

$$\frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} \le \frac{(\rho_g + \varepsilon)\log M(r, f) + O(1)}{\log^{[2]} M(\exp(r^{\rho_g}), f)}$$

$$= \frac{(\rho_g + \varepsilon)\log M(r, f) + O(1)}{r^{\rho_f}} \frac{r^{\rho_g}}{\log^{[2]} M(\exp(r^{\rho_g}), f)}\ r^{\rho_f - \rho_g} \tag{7}$$

As $\varepsilon > 0$ is arbitrary and $\rho_g > \rho_f$ from (7) we get that

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} = 0.$$

Remark 3. *If we replace f by g in the denominator of Theorem 5 then Theorem 5 is still valid which is evident from the following theorem.*

Theorem 6. *Let $f(z)$ and $g(z)$ be two entire functions of finite order such that $0 < \lambda_f$.If $\sigma_g < \infty$ then*

$$(i) \limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), g)} < \infty \text{ if } n \text{ is even.}$$

(ii) Further if $\rho_g > \rho_f$, $\sigma_f < \infty$ and n is odd then

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), g)} = 0.$$

Proof of Theorem 6 is similar to Theorem 5 and so is omitted.

Theorem 7. *Let $f(z)$ and $g(z)$ be two entire functions of finite order such that $0 < \lambda_f$.If $\sigma_g = \infty$ then*

$$(i) \limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} = \infty \text{ if } n \text{ is even.}$$

(ii) Further if $\rho_g < \rho_f$, $\sigma_f = \infty$ and n is odd then

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} = \infty.$$

Proof. (i) As n is even from Lemma 2 we get for all sufficiently large values of r that

$$\frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} \geq \frac{(\lambda_f - \varepsilon)\log M\left(\frac{r}{2^{n-1}}, g\right) + O(1)}{\log^{[2]} M(\exp(r^{\rho_g}), f)}$$

$$= \frac{(\lambda_f - \varepsilon)\log M\left(\frac{r}{2^{n-1}}, g\right) + O(1)}{\left(\frac{r}{2^{n-1}}\right)^{\rho_g}} \frac{r^{\rho_g}}{\log^{[2]} M(\exp(r^{\rho_g}), f)} \frac{1}{(2^{n-1})^{\rho_g}}. \tag{8}$$

As $\varepsilon > 0$ is arbitrary and $\sigma_g = \infty$ from (8) we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} = \infty.$$

(ii) As n is odd from Lemma 2 we get for all sufficiently large values of r that

$$\frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} \geq \frac{(\lambda_g - \varepsilon)\log M\left(\frac{r}{2^{n-1}}, f\right) + O(1)}{\log^{[2]} M(\exp(r^{\rho_g}), f)}$$

$$= \frac{(\lambda_g - \varepsilon)\log M\left(\frac{r}{2^{n-1}}, f\right) + O(1)}{\left(\frac{r}{2^{n-1}}\right)^{\rho_f}} \frac{r^{\rho_g}}{\log^{[2]} M(\exp(r^{\rho_g}), f)} \frac{1}{(2^{n-1})^{\rho_f}} r^{\rho_f - \rho_g}. \quad (9)$$

As $\varepsilon > 0$ is arbitrary and $\rho_g > \rho_f, \sigma_f = \infty$ from (9) we get that

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), f)} = 0.$$

Remark 4. *Replacing f by g in the denominator of Theorem 7 we obtain the following theorem.*

Theorem 8. *Let $f(z)$ and $g(z)$ be two entire functions of finite order such that $0 < \lambda_f$.If $\sigma_g = \infty$ then*

$$(i) \limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), g)} = \infty \text{ if } n \text{ is even.}$$

(ii) Further if $\rho_g < \rho_f$, $\sigma_f = \infty$ and n is odd then

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, f_n)}{\log^{[2]} M(\exp(r^{\rho_g}), g)} = \infty.$$

Remark 5. *Theorem 5,Theorem 6,Theorem 7 and Theorem 8 improves Theorem 5 of Liao and Yang [6].*

Now we study the comparative growth properties of iteration of two set of entire functions.

Theorem 9. *Let f, g, h and k be four entire functions with finite order such that $\rho_g < \rho_k$ and $\rho_f < \rho_h$.Then*

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, h_n)}{\log^{[n]} M(r, f_n)} = \infty.$$

Proof. Case 1: When n is even.

As n is even from Lemma 2 we get for all sufficiently large values of r that

$$\log^{[n]} M(r, h_n) \geq (\lambda_h - \varepsilon)\log M\left(\frac{r}{2^{n-1}}, k\right) + O(1). \quad (10)$$

Now for a sequence of values of r tending to infinity we obtain from 10 that

$$\log^{[n]} M(r, h_n) \geq (\lambda_h - \varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_k - \varepsilon} + O(1). \quad (11)$$

Now from Lemma 1 we get for all sufficiently large values of r that

$$\begin{aligned}\log^{[n]} M(r, f_n) &\leq (\rho_f + \varepsilon) \log M(r, g) + O(1) \\ &< (\rho_f + \varepsilon)\ r^{\rho_g + \varepsilon}. \end{aligned} \tag{12}$$

From (11) and (12) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[n]} M(r, h_n)}{\log^{[n]} M(r, f_n)} \geq \frac{(\lambda_h - \varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_k - \varepsilon} + O(1)}{(\rho_f + \varepsilon)\ r^{\rho_g + \varepsilon}}. \tag{13}$$

As $\varepsilon > 0$ is arbitrary and $\rho_g < \rho_k$ from (13) we get that

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, h_n)}{\log^{[n]} M(r, f_n)} = \infty.$$

Case 2: When n is odd.

From Lemma 2 we get for all sufficiently large values of r that

$$\log^{[n]} M(r, h_n) \geq (\lambda_k - \varepsilon) \log M\left(\frac{r}{2^{n-1}}, h\right) + O(1). \tag{14}$$

Now for a sequence of values of r tending to infinity we obtain from 14 that

$$\log^{[n]} M(r, h_n) \geq (\lambda_k - \varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_h - \varepsilon} + O(1). \tag{15}$$

Now from Lemma 1 we get for all sufficiently large values of r that

$$\begin{aligned}\log^{[n]} M(r, f_n) &\leq (\rho_g + \varepsilon) \log M(r, f) + O(1) \\ &< (\rho_g + \varepsilon)\ r^{\rho_f + \varepsilon}. \end{aligned} \tag{16}$$

From (15) and (16) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[n]} M(r, h_n)}{\log^{[n]} M(r, f_n)} \geq \frac{(\lambda_k - \varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_h - \varepsilon} + O(1)}{(\rho_g + \varepsilon)\ r^{\rho_f + \varepsilon}}. \tag{17}$$

As $\varepsilon > 0$ is arbitrary and $\rho_f < \rho_h$ from (17) we get that

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, h_n)}{\log^{[n]} M(r, f_n)} = \infty.$$

Hence in both the cases

$$\limsup_{r\to\infty} \frac{\log^{[n]} M(r, h_n)}{\log^{[n]} M(r, f_n)} = \infty.$$

References

[1] J. Clunie, *The composition of entire and meromorphic functions,* Mathematical essays dedicated to A.J.Macintyre, Ohio University Press, (1970), 75 - 92.

[2] R.K.Dutta, *Further Growth of Iterated Entire Functions-I,* Journal of Mathematical Ineqalities, **5(**4) (2011), 533-550.

[3] I. Lahiri, *Growth of composite integral functions*, Indian J. Pure Appl. Math., **20**(9) (1989), 899-907.

[4] I. Lahiri and D.K. Sharma, *Growth of composite entire and Meromorphic functions*, Indian J. Pure Appl. Math., **26**(5) (1995),451-458.

[5] B.K.Lahiri and D.Banerjee, *Relative fix points of entire functions*, J. Indian Acad. Math., **19**(1) (1997), 87-97.

[6] I. Liao and C.C. Yang, *On the growth of composite entire functions*, Yokohama Math. J., **46** (1999), 97-107.

[7] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., 69 (1963), 411-414.

[8] A.P. Singh, *Growth of composite entire functions*, Kodai Math. J., **8** (1985), 99-102.

[9] A.P. Singh and M.S. Baloria, *On maximum modulus and maximum term of composition of entire functions*, Indian J. Pure Appl. Math.,**22**(12) (1991), 1019-1026.

[10] G.D. Song and C.C. Yang, *Further growth properties of composition of entire and meromorphic functions,* Indian J. Pure Appl. Math., **15**(1) (1984), 67- 82.

Arkojyoti Biswas
Ranaghat Yusuf Institution
Ranaghat, Nadia-741201
West Bengal, India.
email: *arkojyotibiswas@gmail.com*