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BERWALD MANIFOLDS WITH PARALLEL $S-$STRUCTURES

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Abstract. In this paper we study some rigidity properties for generalized symmetric Berwald manifolds. We prove that any reversible Berwald space with nonzero flag curvature which admits a parallel $s-$structure must be Riemannian.

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1. Introduction

Let (M, F) be a Finsler space, where F is positively homogeneous of degree one. Then we have two ways to define the notion of an isometry of (M, F). On the one hand, we call a diffeomorphism σ of M onto itself an isometry if $F(d\sigma_x(y)) = F(y)$, for any $x \in M$ and $y \in T_xM$. On the other hand, we can also define an isometry of (M, F) to be a one-to-one mapping of M onto itself which preserves the distance of each pair of points of M. It is well known that the two definitions are equivalent if the metric F is Riemannian. The equivalence of these two definitions in the general Finsler case is a result of S. Deng and Z. Hou [3]. Using these result, they proved that the group of isometries $I(M, F)$ of a Finsler space (M, F) is a Lie transformation group of M and for any point $x \in M$, the isotropic subgroup $I_x(M, F)$ is a compact subgroup of $I(M, F)$. These results are important to study homogenous Finsler spaces.

The definition of symmetric Finsler space is a natural generalization of E. Cartan's definition of Riemmanian symmetric spaces. We call a Finsler space (M, F) a symmetric Finsler space if for any point $p \in M$ there exists an involutive isometry s_p of (M, F) such that p is an isolated fixed point of s_p [4, 8].

If we drop the involution property in the definition of symmetric Finsler space, we get a bigger class of Finsler manifolds as symmetric Finsler space.
Let (M, F) be a connected Berwald space. An isometry s_x of (M, F) for which $x \in M$ is an isolated fixed point will be called a symmetry of M at x.

An $s-$structure on (M, F) is a family $\{s_x | x \in M\}$ of symmetries of (M, F). The corresponding tensor field S of type (1,1) defined by $S_x = (s_{x*})_x$ for each $x \in M$ is called the symmetry tensor field of $s-$structure ([7], [9]).

Definition 1. *An $s-$structure $\{s_x | x \in M\}$ on a Berwald space (M, F) is said to be regular if it satisfies the rule*

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y)$$

for every two points $x, y \in M$.

Definition 2. *An $s-$structure $\{s_x\}$ on a Berwald space (M, F) is said to be parallel if the tensor field S is parallel with respect to the Chern connection i.e. $\nabla S = 0$.*

In this paper we study some rigidity properties of Berwald manifolds with parallel $s-$structures:

Theorem 1. *Let (M, F) be a reversible Berwald manifold which admits a parallel $s-$structure. If the flag curvature of (M, F) is everywhere nonzero, then F is Riemannian.*

2. Finsler Spaces

In this section, we give a brief description of basic quantities and fundamental formulas in Finlser geometry, for more details the reader is referred to see [1, 2].

Let M be an $n-$dimensional smooth manifold without boundary and TM denote its tangent bundle. A Finsler structure on M is a map $F : TM \longrightarrow [0, \infty)$ which has the following properties [2]:

1. F is smooth on $\widetilde{TM} := TM - \{0\}$.
2. $F(x, \lambda y) = \lambda F(x, y)$, for any $x \in M, y \in T_xM$ and $\lambda > 0$.
3. F^2 is strongly convex, i.e.,

$$\mathrm{g}_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$$

is positive definite for all $(x, y) \in \widetilde{TM}$.

Let (M, F) be a Finsler $n-$manifold with Finsler function $F : TM \longrightarrow [0, \infty)$. Let $(x, y) = (x^i, y^i)$ be the local coordinates on TM. F is called reversible if $F(x, y) = F(x, -y)$ for any $y \in T_xM$.

Most Finsler quantities are functions on TM rather than M. Some fundamental quantities and relations

$$\mathrm{g}_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y) \quad \text{(fundamental tensor)}$$

$$C_{ijk}(x,y) := \frac{1}{4}\frac{\partial^3 F^2(x,y)}{\partial y^i \partial y^j \partial y^k} \quad \text{(Cartan tensor)}$$

$$\gamma^k_{ij} := \frac{1}{2}\mathrm{g}^{km}\left(\frac{\partial \mathrm{g}_{mj}}{\partial x^i} + \frac{\partial \mathrm{g}_{im}}{\partial x^j} - \frac{\partial \mathrm{g}_{ij}}{\partial x^m}\right)$$

$$N^i_j := \gamma^i_{jk}y^k - C^i_{jk}\gamma^k_{rs}y^r y^s \text{ where } C^i_{jk} = g^{il}C_{ljk}$$

.

According to [2], the pull-back bundle π^*TM admits a unique linear connection, called the Chern connection. Its connection forms are characterized by the structure equations

- Torsion freeness:

$$d\omega^i = \omega^j \wedge \omega^i_j.$$

- Almost $g-$compatibility:

$$dg_{ij} = g_{ik}\omega^k_j + g_{kj}\omega^k_i + 2C_{ijk}\omega^{n+k},$$

where

$$\omega^i := dx^i \qquad \omega^{n+k} := dy^k + y^j\omega^k_j.$$

It is easy to see that torsion freeness is equivalent to the absence of dy^k terms in ω^i_j, namely

$$\omega^i_j = \Gamma^i_{jk}dx^k,$$

together with the symmetry

$$\Gamma^i_{jk} = \Gamma^i_{kj}.$$

To define the flag curvature, we need some differential forms on $TM - \{0\}$. Let

$$\delta y^i = dy^i + N^i_j dx^j.$$

The curvature 2-form of the Chern connection are

$$\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k.$$

Since Ω^i_j are 2-forms on the manifold $TM - \{0\}$, they can be expanded as

$$\Omega^i_j = \frac{1}{2}R^i_{jkl}dx^k \wedge dx^l + P^i_{jkl}dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2}Q^i_{jkl}\frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}.$$

Q vanishes for the Chern connection. Let

$$R_{jikl} = g_{is} R^s_{jkl}.$$

A flag on M at $x \in M$ is a pair (P, y), where P is a plane in the tangent space $T_x M$ and y is a non-zero vector in P. The flag curvature of the flag (P, y) is defined to be

$$K(P, y) := \frac{u^i (y^j R_{jikl} y^l) u^k}{g_y(y, y) g_y(u, u) - [g_y(y, u)]^2},$$

where $u = u^i \frac{\partial}{\partial x^i}$ is any nonzero vector in P such that $P = \mathrm{span}\{y, u\}$. It can be shown that the quantity is independent of the selection of u [2].

Definition 3. *A Finsler metric F on a manifold M is called a Berwald metric if in any standard local coordinate system (x^i, y^i) in $TM - \{0\}$, the Christoffel symbols $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ are functions of $x \in M$ only, in which case, $G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x$.*

3. Generalized symmetric Berwald spaces

A diffeomorphism, $\varphi : M \longrightarrow M$, is an isometry on a Finsler manifold (M, F) if it preserves the Finsler function:

$$F(\varphi(x), d\varphi_x(X)) = F(x, X) \qquad \forall x \in M, X \in T_x M.$$

By the classical Dantzing-van der Waereden Theorem ([6]vol I, chapter I, Theorem 4.7) and the Montgomery- Zippin Theorem ([6],vol I, chapter I, Theorem 4.6), the group of isometries on a connected Finsler manifold form a Lie group. Strictly speaking, these theorems prove the statement for absolute homogeneous Finsler functions. For positive homogeneous Finsler functions consider the metric, d^*, defined by the function

$$F^*(X) = F(X) + F(-X)$$

Then the G is a closed subgroup of G^* defined for d^*. Thus both groups are Lie groups [10].

Let (M, F) be a Berwald space, $p \in M$. Then there exists a neighborhood N_0 of the origin of the tangent space $T_p M$ such that the exponential mapping $\exp_p$ is C^∞ diffeomorphism of N_0 on to a neighborhood N_p of p in M [2]. We can also assume that $N_0 = -N_0$. Now we define a mapping of N_p onto itself by

$$\sigma_p : \exp(y) \longrightarrow \exp(-y) \qquad y \in N_0.$$

Then σ_p is called the geodesic symmetry with respect to p. M is called locally geodesic symmetry if for any $p \in M$, there exists N_p such that σ_p is an isometry of N_p.
Since any isometry of (M, F) is an affine transformation with respect to the connection of F, we see that a locally geodesic symmetric Berwald space (M, F) must be locally symmetric. The definition of globally symmetric Finsler space is a natural generalization of É. Cartan's definition of Riemannian globally symmetric spaces.

Definition 4. *A connected Finsler space (M, F) is said to be symmetric if to each $p \in M$ there is associated an isometry $\sigma_p : M \longrightarrow M$ which is*

(i) *involutive (σ_p^2 is the identity).*

(ii) *has p as an isolated fixed point, that is, there is a neighborhood U of p in which p is the only fixed point of σ_p.*

σ_p is called the symmetry of the point p.

As p is an isolated fixed point of σ_p it follows that $(d\sigma_p)_p = -id$, and therefore symmetric Finsler spaces have reversible metrics and geodesics.
Let (M, F) be a connected symmetric Finsler space, Then (M, F) is (forward-backward) complete and homogeneous that is the group of isometries of (M, F) acts transitively on M [8], [5].

Let (M, F) be a symmetric Finsler space. Then (M, F) is a Berwald space. Furthermore, the connection of F coincides with the Levi-Civita connection of a Riemannian metric g such that (M, g) is a Riemannian symmetric space.

Let (M, F) be a connected Berwald space. An isometry s_x of (M, F) for which $x \in M$ is an isolated fixed point will be called a symmetry of M at x. Clearly, if s_x is a symmetry of (M, g) at x, then the tangent map $S_x = (s_{x*})_x$ has no invariant vector.

An $s-$structure on (M, F) is a family $\{s_x | x \in M\}$ of symmetries of (M, F). The corresponding tensor field S of type (1,1) defined by $S_x = (s_{x*})_x$ for each $x \in M$ is called the symmetry tensor field of $s-$structure.[7], [9]
An $s-$structure $\{s_x | x \in M\}$ is called of order k ($k \geq 2$) if $(s_x)^k = id$ for all $x \in M$ and k is the least integer of this property. Obviously a Berwald space is symmetric if and only if it admits an $s-$structure of order 2.

Definition 5. *An $s-$structure $\{s_x | x \in M\}$ on a Berwald space (M, F) is said to be regular if it satisfies the rule*

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y)$$

for every two points $x, y \in M$.

Similar to the Riemmanian case an $s-$structure $\{s_x\}$ on a connected Berwald space (M, F) is regular if and only if the tensor field S is invariant with respect to all symmetries s_x, i.e.

$$s_{x*}(S) = S, \qquad x \in M \tag{1}$$

Definition 6. *An $s-$structure $\{s_x\}$ on a Berwald space (M, F) is said to be parallel if the tensor field S is parallel with respect to the Chern connection i.e. $\nabla S = 0$.*

Theorem 2. *Each parallel $s-$structure on a Berwald space is regular*

Proof. Suppose $\{s_x\}$ to be a parallel $s-$structure on (M, F). Let $p \in M$ be a fixed point and put $S' = s_{p*}(S)$. Because $\nabla S = 0$ and s_p is connection preserving, we have $\nabla S' = 0$. Now $S'_p = (s_{p*})_p(S_p) = S_p$, from the uniqueness of a parallel extension we have $S' = S$. Thus for all points $p \in M$ we get $(s_{p*})(S) = S$ and hence $\{s_x\}$ is regular.

Proof of theorem 1.

Let (M, F) be a Berwald space and let $\{s_x\}$ be a parallel $s-$structure on (M, F). Let $X, Y, Z \in T_pM$ be tangent vectors and $\omega \in T_p^*M$ a covector at $p \in M$. By parallel translation along each geodesic through p, X, Y, Z, ω can be extended to local vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{\omega}$ with vanishing covariant derivatives at p. Because S is parallel, the local vector fields $S\widetilde{X}, S\widetilde{Y}, S\widetilde{Z}, S^{*-1}\widetilde{\omega}$ have also vanishing covariant derivative at p. Now, because R is invariant with respect to the affine transformation $s_x, x \in M$, we have

$$R(\widetilde{\omega}, \widetilde{X}, \widetilde{Y}, \widetilde{Z}) = R(S^{*-1}\widetilde{\omega}, S\widetilde{X}, S\widetilde{Y}, S\widetilde{Z}) \tag{2}$$

$$\nabla R(\omega, X, Y, Z, U) = \nabla R(S^{*-1}\omega, SX, SY, SZ, SU) \tag{3}$$

Differentiating covariantly (2) in the direction of SU at p and using (3) we get

$$\nabla R(\omega, X, Y, Z, SU) = \nabla R(S^{*-1}\omega, SX, SY, SZ, SU) = \nabla R(\omega, X, Y, Z, U).$$

Thus

$$(\nabla R)_p(\omega, X, Y, Z, (I - S)U) = 0,$$

for all $\omega \in T_p^*M$, $X, Y, Z, U \in T_pM$ and because $(I - S)_p$ is non-singular transformation, we obtain $(\nabla R)_p = 0$. This holds for all $p \in M$ and hence $\nabla R = 0$. So the geodesic symmetry σ_p is an affine symmetry. Now let $q \in M$. Join q to p by a curve γ. Let τ denote the parallel transformation from p to q along γ. Then for any $U, V \in T_qM$, we have

$$g_Y(U, V) = g_{\tau(Y)}(\tau(U), \tau(V)) = g_{d\sigma_p(\tau(Y))}(d\sigma_p(\tau(U)), d\sigma_p(\tau(V))).$$

Now $\tau(Y)$, $\tau(U)$, $\tau(V)$ is the result of the parallel displacement along γ of Y, U, V, respectively. Since σ_p, being an affine symmetry, transforms vectors that are parallel along γ into vectors that are parallel along $\sigma_p(\gamma)$. Therefore $d\sigma_p(\tau(Y))$, $d\sigma_p(\tau(U))$, $d\sigma_p(\tau(V))$ must be the result of parallel displacement along $\sigma_p(\gamma)$ of $d\sigma_p(Y)$, $d\sigma_p(U)$, $d\sigma_p(V)$, respectively. Thus

$$g_{d\sigma_p(\tau(Y))}(d\sigma_p(\tau(U)), d\sigma_p(\tau(V))) = g_{d\sigma_p(Y)}(d\sigma_p(U), d\sigma_p(V)).$$

Therefore

$$g_Y(U, V) = g_{d\sigma_p(Y)}(d\sigma_p U, d\sigma_p V).$$

Thus

$$\begin{aligned} F(d\sigma_p(Y)) &= \sqrt{g_{d\sigma_p(Y)}(d\sigma_p(Y), d\sigma_p(Y))} \\ &= \sqrt{g_Y(Y, Y)} \\ &= F(Y). \end{aligned}$$

So the geodesic symmetry σ_p is an local isometry. So (M, F) is locally geodesic symmetric space. By the assumption, the flag curvature of (M, F) is everywhere non zero. Therefore by Theorem 8.9 of [5] we conclude that F is a Riemmanian metric.

Corollary 3. *If a Berwald space (M, F) admits a parallel $s-$structure then it is locally affine symmetric.*

Corollary 4. *If a Berwald space (M, F) admits a parallel $s-$structure and F is reversible then it is locally geodesic symmetric.*

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References

[1] P. L. Antonelli , R. S. Ingarden and M. Matsumato, *The Theory of sprays and Finsler spaces with applications in Physics and Biology*, FTPH vol. 58, Kluwer, Dordecht 1993.

[2] D. Bao, S. S. Chern, Z. Shen, *An introduction to Riemann-Finsler Geometry* , Springer-Verlag, New York, 2000.

[3] S. Deng, Z. Hou, *The group of isometries of a Finsler space*, Pacific. J. Math. 207, 1 (2002), 149-155.

[4] S. Deng, Z, Hou, *Invariant Finsler metrics on homogeneous manifolds*, Journal of Physics A: Math. Gen. 37 (2004), 8245-8253.

[5] S. Deng, Z. Hou, *On symmetric Finsler spaces, Israel Journal of Mathematics*, 162(2007), 197-219.

[6] S. Kobayashi, K. Nomizu, *Foundation of differential geometry I. II.*, John Wily and Sons (1963)(1969).

[7] O, Kowalski, *Generalized symmetric spaces, Lect.* Notes in Math. Springer Verlag, 1980.

[8] D. Latifi, A. Razavi, *On homogeneous Finsler spaces*, Rep. Math. Phys. 57 (2006), 357-366. Erratum: Rep. Math. Phys, 60 (2007) 347.

[9] A. J. Ledger, M. Obata, *Affine and Riemannian s-manifolds*, J. Differential Geometry 2 (1968), 451-459.

[10] Z. I. Szabo, *Berwald metrics constructed by Chevalley's polynomials*, arxiv:math/0601522v2, 2008.

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