

**APPLICATIONS OF THE ROPER-SUFFRIDGE EXTENSION
OPERATOR TO THE SPIRALLIKE MAPPINGS OF TYPE β**

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ABSTRACT. The Roper-suffridge extension operator extends a locally univalent mapping defined on the unit disk of \mathbb{C} to the locally biholomorphic mapping defined on the Euclidean unit ball of \mathbb{C}^n . Also, the extension of one variable mapping that is either convex or starlike has the same property in several variables. In the present paper, we consider certain generalizations of the operator and we examine the condition under which this operator will take a spirallike mapping of type β of the unit disc to a mapping of the same type defined on the unit ball.

2000 Mathematics Subject Classification: 30C45.

Keywords: Roper-Suffridge extension operator, Biholomorphic mapping, Spirallike mapping of type β .

1. INTRODUCTION

Let \mathbb{C}^n be the vector space of n -complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k$ and Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n by B^n . In the case of one complex variable, B^1 is denoted by U . It is convenient, if $n \geq 2$, to write a vector $z \in \mathbb{C}^n$ as $z = (z_1, \hat{z})$, where $z_1 \in \mathbb{C}$ and $\hat{z} \in \mathbb{C}^{n-1}$.

Let Q_n denote the set of all homogenous polynomials $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ of degree 2. That is, $Q(\lambda z) = \lambda^2 Q(z)$ for all $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. We know that Q_n is a Banach space with the norm

$$\|Q\| = \sup_{z \in \mathbb{C}^n \setminus \{0\}} \frac{|Q(z)|}{\|z\|^2}, \quad Q \in Q_n.$$

Let $H(B^n, \mathbb{C}^n)$ denote the topological vector space of all holomorphic mappings $F : B^n \rightarrow \mathbb{C}^n$. If $F \in H(B^n)$, we say that F is normalized if $F(0) = 0$ and $DF(0) = I$, where DF is the Fréchet differential of F and I is the identity operator

on \mathbb{C}^n . Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n , and $S_1 = S$ is the classical family of univalent mappings of U .

A map $f \in S(B^n)$ is said to be convex if its image is convex domain in \mathbb{C}^n , and starlike if the image is a starlike domain with respect to 0. We denote the classes of normalized convex and starlike mappings on B^n respectively by $K(B^n)$ and $S^*(B^n)$.

In 1995, Roper and Suffridge [4] introduced an extension operator which gives a way of extending a (locally) univalent function on the unit disc to a (locally) univalent mapping of B^n into \mathbb{C}^n .

For fixed $n \geq 2$, the Roper-Suffridge extension operator (see [2] and [4]) in the function

$$[\Phi_n(f)](z) = (f(z_1), \sqrt{f'(z_1)}\hat{z}), \quad f \in S_1, z \in B^n$$

The branch of the power function is chosen so that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

The following results illustrate the important and usefulness of the Roper-Suffridge extension operator

$$\Phi_n(K_1) \subseteq K_n, \quad \Phi_0(S_1^*) \subseteq S_n^*.$$

The first was proved by K. A. Roper and T. J. Suffridge when they introduced their operator [4], while the second result was given by I. Graham and G. Kohr [1]. Until now, it is difficult to constant the concrete convex mappings, starlike mappings on B^n . By making use of the Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator. A good treatment of further applications of the Roper-Suffridge extension operator can be found in the recent book by Graham and Kohr [2].

Definition 1. Let $f : B^n \rightarrow \mathbb{C}^n$ be a normalized locally biholomorphic mapping and let $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear operator such that $Re \langle A(z), z \rangle > 0$ for $z \in \mathbb{C}^n \setminus \{0\}$. We say that f is spirallike with respect to A if

$$Re \langle [Df(z)]^{-1}Af(z), z \rangle \geq 0, \quad z \in B^n \setminus \{0\}.$$

In the case that $A = e^{i\beta}I_n$, where $|\beta| < \frac{\pi}{2}$, we say that f is spirallike of type β .

Then for $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$, a mapping $f \in S_n(B^n)$ is said to be spirallike mapping of type β if

$$Re\{e^{-i\beta}\langle DF^{-1}(z)F(z), z \rangle\} \geq 0.$$

Note that any spirallike mapping with respect to a linear operator A such that $Re \langle A(z), z \rangle > 0$ for $z \in \mathbb{C}^n \setminus \{0\}$ is biholomorphic (see [6]). We denote by $\hat{S}_\beta(B^n)$ the class of spirallike mappings of type β , $|\beta| < \frac{\pi}{2}$. Also, we have

$$f \in \hat{S}_\beta(B^n) \iff f \in S(U) \text{ and } Re\{e^{-i\beta} \frac{f(\xi)}{\xi f'(\xi)}\} \geq 0 \text{ for } \xi \in U.$$

For a function $f \in S$, we introduce the quantity

$$\Lambda_f(z) = \frac{1 - |z|^2}{2} \frac{f''(z)}{f'(z)} - \bar{z}, \quad (1)$$

for $z \in U$. Now, fix $z \in U$. The disk automorphism transform is denoted by ψ . In other words,

$$\psi(w) = \frac{z - w}{1 - \bar{z}w}.$$

Consider the Koebe transform of f with respect to disk automorphism ψ by the form

$$g(w) = \frac{f(\psi(w)) - f(\psi(0))}{f'(\psi(0))\psi'(0)}$$

for $w \in U$. Clearly, $g \in U$, and a simple calculation shows that $g''(0) = -2\Lambda_f(z)$. It then follows that g has a power series expansion of the form

$$g(w) = w - \Lambda_f(z)w^2 + \mathcal{O}(\|w\|^3),$$

for $w \in U$. The well-known coefficient bound for the second coefficient of a function in S gives

$$|\Lambda_f(z)| \leq 2,$$

for $f \in S$ and $z \in U$.

Definition 2. Let $Q \in Q_{n-1}$. For any $f \in S$, define the operator $\Phi_Q(f) : B^n \rightarrow \mathbb{C}^n$ by

$$[\Phi_Q(f)](z) = (f(z_1) + f'(z_1)Q(\hat{z}), \sqrt{f'(z_1)}\hat{z}), \quad z = (z_1, \hat{z}) \in B^n,$$

we choose the branch of the power function such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

Recently Muir [3] proved the following results:

Theorem 1. If $Q \in Q_{n-1}$, then $\Phi_Q(K) \subseteq K(B^n)$ if and only if $\|Q\| \leq \frac{1}{2}$.

Theorem 2. If $Q \in Q_{n-1}$, then $\Phi_Q(S^*) \subseteq S^*(B^n)$ if and only if $\|Q\| \leq \frac{1}{4}$.

In this paper, we shall see that if $Q \in Q_{n-1}$, then $\Phi_Q(\hat{S}_\beta) \subseteq \hat{S}_\beta(B^n)$ if and only if $\|Q\| \leq \frac{1}{4}$.

2. SOME LEMMAS

In order to prove the main results, we need the following lemmas.

Lemma 3. (see [7]). *Let p be a holomorphic function on U . If $\operatorname{Re} p(z) > 0$ and $p(0) > 0$, then*

$$|p'(z)| \leq \frac{2\operatorname{Re} p(z)}{1 - |z|^2}. \quad (2)$$

Lemma 4. (see [7]). *Let p be a normalized biholomorphic function on U . Then*

$$|(1 - |z|^2) \frac{f''(z)}{f'(z)} - 2\bar{z}| \leq 4. \quad (3)$$

3. MAIN RESULTS

Theorem 5. *Let $n \geq 2$ and $Q \in \mathcal{Q}_{n-1}$. Then $\Phi_Q(\hat{S}_\beta) \subseteq \hat{S}_\beta(B^n)$ if and only if $\|Q\| \leq \frac{1}{4}$.*

Proof. Suppose that $f \in \hat{S}_\beta$, and $\|Q\| \leq \frac{1}{4}$, we write $F = \Phi_Q(f)$ for $z \in B^n$. By the simple calculations, we have

$$DF(z) = \begin{pmatrix} f'(z_1) + f''(z_1)Q(\hat{z}) & f'(z_1)DQ(\hat{z}) \\ \frac{1}{2} \frac{f''(z_1)}{\sqrt{f'(z_1)}} \hat{z} & \sqrt{f'(z_1)} I \end{pmatrix}$$

where I is the identity operator on \mathbb{C}^{n-1} . Then we see that

$$DF(z)^{-1} = \begin{pmatrix} \frac{1}{f'(z_1)} & -\frac{DQ(\hat{z})}{\sqrt{f'(z_1)}} \\ -\frac{1}{2} \frac{f''(z_1)}{(f'(z_1))^2} \hat{z} & \frac{1}{\sqrt{f'(z_1)}} I + \frac{1}{2} \frac{f''(z_1)DQ(\hat{z})}{(f'(z_1))^{3/2}} \hat{z} \end{pmatrix}$$

Therefore

$$\begin{aligned} \langle DF(z)^{-1}F(z), z \rangle &= \frac{f(z_1)}{f'(z_1)} \bar{z}_1 - \bar{z}_1 Q(\hat{z}) \\ &\quad + \|\hat{z}\|^2 \left(1 - \frac{1}{2} \frac{f(z_1)f''(z_1)}{[f'(z_1)]^2} + \frac{1}{2} \frac{f''(z_1)}{f'(z_1)} Q(\hat{z}) \right). \end{aligned}$$

We must show that the real part of $e^{-i\beta} \langle DF(z)^{-1}F(z), z \rangle$ is positive for $z \in B \setminus \{0\}$. Now we have two case:

First, if $\hat{z} = 0$, then

$$Re[e^{-i\beta} \langle DF(z)^{-1}F(z), z \rangle] = Re \left(e^{-i\beta} \frac{f(z_1)}{f'(z_1)} \bar{z}_1 \right) = |z_1|^2 Re \left(e^{-i\beta} \frac{f(z_1)}{z_1 f'(z_1)} \right) > 0$$

We therefore assume that $\hat{z} \neq 0$. Obviously, the mapping F is holomorphic at every point $z = (z_1, \hat{z}) \in \bar{B}^n$ with $\hat{z} \neq 0$. Let $z = \xi u$, $u \in \mathbb{C}^n$, $\|u\| = 1$ and $\xi \in \bar{U} \setminus \{0\}$, then we have

$$\begin{aligned} Re[e^{-i\beta} \bar{z} DF(z)^{-1}F(z)] &\geq 0 \Leftrightarrow \\ Re\left[\frac{e^{-i\beta} \bar{z} DF(z)^{-1}F(z)}{\|z\|^2}\right] &\geq 0 \Leftrightarrow \\ Re\frac{e^{-i\beta} \bar{\xi} \bar{u} DF(\xi u)^{-1}F(\xi u)}{\|\xi\|^2} &\geq 0 \Leftrightarrow \\ Re\frac{e^{-i\beta} \bar{u} DF(\xi u)^{-1}F(\xi u)}{\xi} &\geq 0 \end{aligned}$$

Since the expression

$$Re\frac{e^{-i\beta} \bar{u} DF(\xi u)^{-1}F(\xi u)}{\xi}$$

is the real part of a holomorphic function with respect to ξ , it is a harmonic function. By the minimum principle for harmonic functions, we know that it attains its minimum of $\|\xi\| = 1$, so we only need to prove

$$Re e^{-i\beta} \langle DF(z)^{-1}F(z), z \rangle \geq 0, \tag{4}$$

for all $z = (z_1, \hat{z}) \in \partial B^n$, $\hat{z} \neq 0$.

Taking into account (2) and (4), we need to show that

$$\begin{aligned} &Re e^{-i\beta} \left(\frac{f(z_1)}{f'(z_1)} \bar{z}_1 - \bar{z}_1 Q(\hat{z}) + \|\hat{z}\|^2 \left(1 - \frac{1}{2} \frac{f(z_1) f''(z_1)}{[f'(z_1)]^2}\right) + \frac{1}{2} \frac{f''(z_1)}{f'(z_1)} Q(\hat{z}) \right) \\ &= Re e^{-i\beta} \left(\frac{f(z_1)}{z_1 f'(z_1)} \|z_1\|^2 + \|\hat{z}\|^2 \left(1 - \frac{1}{2} \frac{f(z_1) f''(z_1)}{[f'(z_1)]^2}\right) + Q(\hat{z}) \left(\frac{1}{2} \frac{f''(z_1)}{f'(z_1)} \|\hat{z}\|^2 - \bar{z}_1\right) \right) \\ &= Re e^{-i\beta} \frac{f(z_1)}{z_1 f'(z_1)} \|z_1\|^2 + e^{-i\beta} \|\hat{z}\|^2 \left(1 - \frac{1}{2} \frac{f(z_1) f''(z_1)}{[f'(z_1)]^2}\right) + e^{-i\beta} Q(\hat{z}) \Lambda_f(z_1) \geq 0. \end{aligned}$$

Define

$$p(\xi) = e^{-i\beta} \frac{f(\xi)}{\xi f'(\xi)}, \tag{5}$$

then

$$e^{-i\beta} \left(1 - \frac{1}{2} \frac{f(z_1)f''(z_1)}{[f'(z_1)]^2}\right) = \frac{1}{2}e^{-i\beta} + \frac{e^{-i\beta}}{2}(p(\xi) + \xi p'(\xi)), \quad (6)$$

set

$$G(z) = e^{-i\beta} \left(\frac{f(z_1)}{z_1 f'(z_1)}|z_1|^2 + e^{-i\beta} \|\hat{z}\|^2 \left(1 - \frac{1}{2} \frac{f(z_1)f''(z_1)}{[f'(z_1)]^2}\right) + e^{-i\beta} Q(\hat{z})\Lambda_f(z_1)\right). \quad (7)$$

Notice that f is spirallike of type β on the unit disk, hence $Re p(z_1) > 0$ and $p(0) = 1$. By the Lemma 4, we can obtain that

$$|p'(z_1)| \leq \frac{2Re p(z_1)}{1 - |z_1|^2}.$$

Substituting (5) and (6) into (7), we have

$$G(z) = |z_1|^2 p(z_1) + e^{-i\beta} \frac{1 - |z_1|^2}{2} + \frac{1 - |z_1|^2}{2} (p(z_1) + z_1 p'(z_1)) + e^{-i\beta} Q(\hat{z})\Lambda_f(z_1).$$

Hence,

$$\begin{aligned} ReG(z) &\geq |z_1|^2 Re p(z_1) + |e^{-i\beta}| \frac{1 - |z_1|^2}{2} + \frac{1 - |z_1|^2}{2} Re p(z_1) \\ &\quad - \frac{1 - |z_1|^2}{2} |z_1 p'(z_1)| - |e^{-i\beta}| \|Q\| |\Lambda_f(z_1)| \\ &\geq \frac{1 + |z_1|^2}{2} Re p(z_1) + \frac{1 - |z_1|^2}{2} - |z_1| Re p(z_1) - \|Q\| |\Lambda_f(z_1)| \\ &= \frac{(1 - |z_1|)^2}{2} Re p(z_1) + \frac{1 - |z_1|^2}{2} - \|Q\| |\Lambda_f(z_1)|. \end{aligned}$$

Now, we show that the $ReG(z) \geq 0$. Using $|\Lambda_f(z_1)| \leq 2$ and $\|Q\| \leq \frac{1}{4}$, we have

$$ReG(z) \geq \frac{(1 - |z_1|)^2}{2} Re p(z_1) + \frac{1 - |z_1|^2}{2} + \frac{1}{2} \geq \frac{1 - |z_1|^2}{2} \geq 0.$$

Therefore $F \in \hat{S}_n(\beta)$.

For the converse, suppose that $\|Q\| \geq 1/4$, let $f \in \hat{S}_1(\beta)$ be the function

$$f(\xi) = \frac{\xi}{(1 - \xi)^{2e^{-i\beta} \cos \beta}}, \quad (8)$$

for $\xi \in U$, and write $F = \Phi_Q(f)$. Let $u \in \partial B^{n-1}$ be such that $Q(u) = -\|Q\|$, and $z = (r, \sqrt{1-r^2}u) \in \partial B$ for $r \in (0, 1)$. A simple calculation reveals that $\Lambda_f(z_1) = \frac{e^{-i\beta}(1-|z_1|^2)\cos\beta(2-z_1+2z_1e^{-i\beta}\cos\beta)}{(1-z_1)(1-z_1+2z_1e^{-i\beta}\cos\beta)} - \bar{z}_1$, and $Q(\hat{z}) = -(1-r^2)\|Q\|$. Therefore, we have

$$\langle DF(z)^{-1}F(z), z \rangle = \frac{f(z_1)}{f'(z_1)}\bar{z}_1 + \|\hat{z}\|^2\left(1 - \frac{1}{2} \frac{f(z_1)f''(z_1)}{[f'(z_1)]^2}\right) - \Lambda_f(z_1)(1-r^2)\|Q\|^2 \quad (9)$$

now, by simple calculations, we have

$$\begin{aligned} \langle DF(z)^{-1}F(z), z \rangle &= \frac{|z_1|^2(1-z_1)}{1-z_1+2z_1e^{-i\beta}\cos\beta} \\ &\quad + \|\hat{z}\|^2 \left(1 - \frac{z_1e^{-i\beta}\cos\beta(2-z_1+2z_1e^{-i\beta}\cos\beta)}{(1-z_1+2z_1e^{-i\beta}\cos\beta)^2}\right) \\ &\quad - \left(\frac{e^{-i\beta}(1-|z_1|^2)\cos\beta(2-z_1+2z_1e^{-i\beta}\cos\beta)}{(1-z_1)(1-z_1+2z_1e^{-i\beta}\cos\beta)} - \bar{z}_1\right) (1-r^2)\|Q\|^2 \\ &= \frac{r^2(1-r^2)}{(1+r)(1-r+2re^{-i\beta}\cos\beta)} \\ &\quad + (1-r^2) \left(\frac{(1-r)^2+2re^{-i\beta}\cos\beta(re^{-i\beta}\cos\beta+1)}{(1-r+2re^{-i\beta}\cos\beta)^2}\right) \\ &\quad + (1-r^2) \left(\frac{-3r^2e^{-i\beta}\cos\beta}{(1-r+2re^{-i\beta}\cos\beta)^2}\right) \\ &\quad - \frac{e^{-i\beta}\cos\beta(2-r-3r+2r(r+1)e^{-i\beta}\cos\beta)+r(r-1)}{1-r+2re^{-i\beta}\cos\beta} \|Q\|(1-r^2), \end{aligned}$$

then

$$\begin{aligned} \operatorname{Re} e^{-i\beta} \langle DF(z)^{-1}F(z), z \rangle &= \left[\frac{r^2(1+r)^2}{(1-r^2)^2+4r(1+r)^2\cos^2\beta} \right. \\ &+ \frac{1-r^2+r\cos^2\beta(2r\cos^2\beta+2-2r\sin^2\beta-3r)((1-r)^2+4r(1-r)\cos\beta+4r^2\cos^3\beta)}{((1-r)^2\cos\beta+4r(1-r)\cos\beta+4r^2\cos^3\beta)^2+\sin^2\beta((1-r)^2-4r^2\cos^2\beta)^2} \\ &+ \frac{r\sin^2\beta\cos\beta(3r-2-4r\cos^2\beta)((1-r)^2-4r^2\cos^2\beta)}{((1-r)^2\cos\beta+4r(1-r)\cos\beta+4r^2\cos^3\beta)^2+\sin^2\beta((1-r)^2-4r^2\cos^2\beta)^2} \\ &- \left(\frac{2(1+r)\cos^3\beta+r(1+r)\cos^2\beta(\cos\beta(r-2)+3r+2)}{1+r^2+2r\cos 2\beta} \right. \\ &\left. + \frac{2r(1-r^2)(\sin\beta\cos\beta)^2(1+2\cos\beta)+2\sin^2\beta\cos\beta(1-r)(r-2r^2-1)}{1+r^2+2r\cos 2\beta} \right) \|Q\| (1-r^2). \end{aligned}$$

For r sufficiently close to 1 and $\|Q\| \geq 1/4$, the above relation will be negative, proving that $F \notin \hat{S}_n(\beta)$. This complete the proof.

Set $\beta = 0$, then we get the following corollary(this result generalized the result of J. R. Muir [3]).

Corollary 6. *Let $n \geq 2$ and $Q \in Q_{n-1}$. Then $\Phi_Q(S_1^*) \subseteq S_n^*$ if and only if $\|Q\| \leq \frac{1}{4}$.*

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