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ON η-EINSTEIN LP-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study η-Einstein LP-Sasakian manifolds.

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1. Introduction

In 1989 Matsumoto [7] introduced the notion of Lorentzian para-Sasakian manifolds. Then Mihai and Rosca [5] defined the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [8], Matsumoto, Mihai and Rosca [9], De and Shaikh [13], Ozgur [4] and many others.
The Ricci tensor S of an LP-Sasakian manifold is said to be η-Einstein if its Ricci tensor satisfies the following condition

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{1}$$

where a, b are smooth functions.
η-Einstein LP-Sasakian manifolds have been studied by Mihai, Shaikh and De [6]. Also Shaikh, De and Binh [2] studied K-contact η-Einstein manifolds satisfying certain curvature conditions. Example of an η-Einstein manifold is given by Okumura [10]. Motivated by the above works we study some properties of η-Einstein LP-Sasakian manifolds. The paper is organized as follows:
In section 2, some preliminary results are recalled. After preliminaries, we find out the significance of the associated scalars in an LP-Sasakian η-Einstein manifold. In the next Section, we prove that the functions a and b of the defining equation (1) are constants, provided $tr\phi = 0$. We also obtain a necessary and sufficient condition for an LP-Sasakian manifold to be an η-Einstein manifold. Finally, we cited some examples of η-Einstein LP-Sasakian manifolds.

2. Preliminaries

Let M^n be an n-dimensional differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p\,\varepsilon M$, the tensor g_p: $T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,.....,+)$, where T_pM denotes the tangent space of M at p and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2}$$

$$g(X,\xi) = \eta(X), g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{3}$$

for all vector fields X, Y. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold M^n with the structure (ϕ, ξ, η, g) is called Lorentzian almost paracontact manifold [7]. In the Lorentzian almost paracontact manifold M^n, the following relations hold [7] :

$$\phi\xi = 0, \eta(\phi X) = 0, \tag{4}$$

$$\Omega(X,Y) = \Omega(Y,X), \tag{5}$$

where $\Omega(X,Y) = g(X, \phi Y)$.

Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor S and the scalar curvature r are defined by

$$S(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1$.

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called Lorentzian paracontact manifold if

$$\Omega(X,Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}.$$

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called an LP-Sasakian manifold [7] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In an LP-Sasakian manifold the 1- form η is closed. Also in [7], it is proved that if an n- dimensional Lorentzian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1- form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then M^n admits an LP-Sasakian structure. Further, on such an LP-Sasakian manifold M^n (ϕ, ξ, η, g), the following relations hold [7]:

$$\eta(R(X,Y)Z) = [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{6}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{7}$$

$$R(X,Y)\xi = [\eta(Y)X - \eta(X)Y], \tag{8}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{9}$$

$$(\nabla_X \phi)(Y) = [g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{10}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold. Also since the vector field η is closed in an LP-Sasakian manifold, we have ([8],[7])

$$(\nabla_X \eta)Y = \Omega(X,Y), \tag{11}$$

$$\Omega(X,\xi) = 0, \tag{12}$$

$$\nabla_X \xi = \phi X, \tag{13}$$

for any vector field X and Y.

3. Significance of the associated scalars in an LP-Sasakian η-Einstein manifold

We can express (1) as follows:

$$S(X,\xi) = (a-b)g(X,\xi). \tag{14}$$

From (14), we conclude that $(a-b)$ is an eigen value of the Ricci operator Q defined by $S(X,Y) = g(QX,Y)$ and ξ is an eigen vector corresponding to this eigen value.

Let V be any other vector orthogonal to ξ so that

$$\eta(V) = 0. \tag{15}$$

From (1), we obtain

$$S(X,V) = ag(X,V) + b\eta(X)\eta(V), \tag{16}$$

Hence in virtue of (15), we get

$$S(X,V) = ag(X,V). \tag{17}$$

From (17), we see that a is an eigen value of the Ricci operator Q and V is an eigen vector corresponding to this eigen value. If the manifold under consideration is n-dimensional and V is any vector orthogonal to ξ, it follows from a known result in linear algebra [12] that the eigen value a is of multiplicity $(n-1)$. Hence the multiplicity of the eigen value $(a-b)$ must be 1. Therefore we can state the following:

Theorem 3.1. In an LP-Sasakian η-Einstein manifold of dimension n, the Ricci operator Q has only two distinct eigen values $(a-b)$ and a of which the former is simple and the later is of multiplicity $(n-1)$.

4. η-Einstein manifolds

This section deals with η-Einstein LP-Sasakian manifolds.

From (1) we have

$$S(\phi X,Y) = ag(\phi X,Y), \tag{18}$$

$$S(\xi,\xi) = -a + b. \tag{19}$$

Theorem 4.1. The Ricci curvature of an η-Einstein LP-Sasakian manifold in the direction of ξ is equal to $-(n-1)$.

Proof. Substituting ξ for X in (7) we have the theorem.

Theorem 4.2. The functions a and b of the defining equation (1) are constants, provided tr $\phi = 0$.

Proof. Equation (19) and (7) imply

$$-a+b=1-n. \tag{20}$$

So we need only to show that a is constant. Taking a frame field we get from (1),

$$\sum_{i=1}^{n}\epsilon_i S(e_i,e_i)=a\sum_{i=1}^{n}\epsilon_i g(e_i,e_i)+b\sum_{i=1}^{n}\epsilon_i\eta(e_i)\eta(e_i),$$

which gives

$$r=na-b,$$

where r is the scalar curvature of the manifold. Now differentiating the above equation we have

$$dr(X)=nda(X)-db(X)=(n+1)da(X). \tag{21}$$

Again from (1) we have

$$QX=aX+b\eta(X)\xi. \tag{22}$$

Differentiating (22) along Y, we get

$$(\nabla_Y Q)X=(Ya)X+(Yb)\eta(X)\xi+bg(\phi X,Y)\xi+b\eta(X)\phi Y. \tag{23}$$

Contracting the above equation with respect to Y, we get

$$(divQ)X=Xa+(\xi b)\eta(X)+b\eta(X)tr\phi. \tag{24}$$

Using the identity [11] $(divQ)X=\frac{dr(X)}{2}$, (21) and tr$\phi=0$, we get

$$(n-1)da(X)=2db(\xi)\eta(X). \tag{25}$$

Putting $X=\xi$ in it, we get

$$(n-1)da(\xi)=-2db(\xi)=2da(\xi),$$

which gives $da(\xi)=0$ and hence $db(\xi)=0$. Consequently (25) yields $da(X)=0$.

We now obtain a necessary and sufficient condition for an LP-Sasakian manifold to be an η-Einstein manifold. In an LP-Sasakian manifold, the following relation holds [1]

$$
\begin{aligned}
R(X,Y)\phi Z &= \phi R(X,Y)Z + g(Y,Z)\phi X \\
&\quad - g(X,Z)\phi Y + g(X,\phi Z)Y - g(Y,\phi Z)X \\
&\quad + 2[g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)]\xi \\
&\quad + 2[\eta(Y)\phi X - \eta(X)\phi Y]\eta(Z). \qquad (26)
\end{aligned}
$$

Taking a frame field and contracting (26) with respect to X, we get

$$
\begin{aligned}
S(Y,\phi Z) &= (C_1^1\overline{R})(Y,Z) \\
&\quad + [g(Y,Z) + 2\eta(Y)\eta(Z)]tr\phi - (n+1)g(Y,\phi Z), \qquad (27)
\end{aligned}
$$

where C_1^1 denotes contraction at the first slot and $\overline{R} = \phi R$.
Since $(C_1^1\overline{R})(Y,Z) = (C_1^1\overline{R})(Z,Y)$, from the above it is obvious that

$$S(Y,\phi Z) = S(Z,\phi Y). \qquad (28)$$

Theorem 4.3. In order that an LP-Sasakian manifold to be an η-Einstein manifold it is necessary and sufficient that the symmetric tensor $(C_1^1\overline{R})$ and Ω should be linearly dependent, provided $tr\phi = 0$.

Proof. At first we assume that $(C_1^1\overline{R})$ and Ω are linearly dependent. Then from (27) we have

$$S(Y,\phi Z) = \lambda g(Y,\phi Z),$$

where λ is a scalar. Now using Theorem 4.2 we can easily seen that the manifold is a η-Einstein manifold.
Conversely, let the manifold is an η-Einstein manifold. Then we have

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z).$$

Replacing Y by ϕY in the above equation we obtain

$$S(Y,\phi Z) = ag(Y,\phi Z). \qquad (29)$$

Using (29) in (27) we see that $(C_1^1\overline{R})$ and Ω are linearly dependent.

5. Examples

Example 5.1: [14] A conformally flat LP-Sasakian manifold is an η-Einstein manifold.

Example 5.2: [4] A ϕ-conformally flat LP-Sasakian manifold is an η-Einstein manifold.

Example 5.3: Let $(M^{n-1}, \widetilde{g})$ be a hypersurface of (M^n, g). If A is the (1,1) tensor corresponding to the normal valued second fundamental tensor H, then we have ([3],p.41),

$$\widetilde{g}(A_\xi(X), Y) = g(H(X,Y), \xi) \tag{30}$$

where ξ is the unit normal vector field and X, Y are tangent vector fields.
Let H_ξ be the symmetric (0,2)tensor associated with A_ξ in the hypersurface defined by

$$\widetilde{g}(A_\xi(X), Y) = (H_\xi(X,Y). \tag{31}$$

A hypersurface of a Riemannian manifold (M^n, g) is called quasi-umbilical ([3], p.147) if its second fundamental tensor has the form

$$H_\xi(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y) \tag{32}$$

where ω is a 1-form, the vector field corresponding to the 1-form ω is a unit vector field, and α, β are scalars. If $\alpha = 0$ (respectively $\beta = 0$ or $\alpha = \beta = 0$) holds, then it is called cylindrical (respectively umbilical or geodesic).
Now from (30), (31) and (32) we obtain

$$g(H(X,Y), \xi) = \alpha g(X,Y) g(\xi,\xi) + \beta\omega(X)\omega(Y) g(\xi,\xi)$$

which implies that

$$H(X,Y) = \alpha g(X,Y)\xi + \beta\omega(X)\omega(Y)\xi, \tag{33}$$

since ξ is the only unit normal vector field.

We have the following equation of Gauss ([3], p.45) for any vector fields X, Y, Z, W tangent to the hypersurface

$$\begin{aligned} g(R(X,Y)Z,W) &= \widetilde{g}(\widetilde{R}(X,Y)Z,W) - g(H(X,W),H(Y,Z)) \\ &\quad + g(H(Y,W),H(X,Z)), \end{aligned} \tag{34}$$

where $\widetilde{R}$ is the curvature tensor of the hypersurface.

Let us assume that the hypersurface is quasi-umbilical. Then from (33) and (34) it follows that

$$\begin{aligned} g(R(X,Y)Z,W) &= \widetilde{g}(\widetilde{R}(X,Y)Z,W) + \alpha^2[g(Y,W)g(X,Z) \\ &\quad -g(X,W)g(Y,Z)] + \alpha\beta[g(Y,W)\omega(X)\omega(Z) \\ &\quad +g(X,Z)\omega(Y)\omega(W) - g(X,W)\omega(Y)\omega(Z) \\ &\quad -g(Y,Z)\omega(X)\omega(W)]. \end{aligned} \tag{35}$$

We know that every LP-Sasakian space form is of constant curvature 1 [14]. Hence we have

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$$

which implies that

$$g(R(X,Y)Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W). \tag{36}$$

Using (36) in (35) we have

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z,W) &= (\alpha^2-1)[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)] \\ &\quad -\alpha\beta[g(Y,W)\omega(X)\omega(Z) + g(X,Z)\omega(Y)\omega(W) \\ &\quad -g(X,W)\omega(Y)\omega(Z) - g(Y,Z)\omega(X)\omega(W)]. \end{aligned} \tag{37}$$

Let $\{e_i\}$, $i = 1,2,.....,n$ be an orthonormal frame at any point of the manifold. Then putting $X = W = \{e_i\}$ in (37) and taking summation over i , we get

$$\begin{aligned} \sum_{i=1}^{n} \epsilon_i \widetilde{g}(\widetilde{R}(e_i,Y)Z,e_i) &= (\alpha^2-1)\sum_{i=1}^{n} \epsilon_i [g(e_i,e_i)g(Y,Z) - g(e_i,Z)g(Y,e_i)] \\ &\quad -\alpha\beta \sum_{i=1}^{n} \epsilon_i [g(Y,e_i)\omega(e_i)\omega(Z) + g(e_i,Z)\omega(Y)\omega(e_i) \\ &\quad -g(e_i,e_i)\omega(Y)\omega(Z) - g(Y,Z)\omega(e_i)\omega(e_i)], \end{aligned}$$

which implies that

$$\widetilde{S}(Y,Z) = [(\alpha^2-1)(n-1) - \alpha\beta]g(Y,Z) + \alpha\beta(n-2)\omega(Y)\omega(Z). \tag{38}$$

Thus a quasi-umbilical hypersurface of an LP-Sasakian space form is η-Einstein.

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