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CANTOR'S INTERSECTION THEOREM AND BAIRE'S CATEGORY THEOREM IN GENERALIZED METRIC SPACE

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Abstract. D-metric space is an interesting nonlinear generalization of metric space which was discovered and studied in details by B.C. Dhage in his Ph.D thesis. In this paper, we establish Cantor's intersection theorem and Baire's category theorem in D-metric spaces using the modified concept of open ball given by R. Asim et al.

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1. Introduction

In [7] Dhage introduced a new structure of a generalized metric space, that is, D-metric space in his Ph.D. thesis [6] in an attempt to obtain similar result of these for metric space. A number of fixed point theorems for nonlinear contraction type mappings have been done in metric space [see for example [2], [3]] and sometimes on some generalized metric structure in [[4], [5]]. Dhage's definition uses the symmetry and tetrahedral axioms present in Gahler's definition for 2-metric space and includes the coincidence axiom that $d(x, y, z) = 0$ iff $x = y = z$. Geometrically, in plane, 2-metric [see [10], [14]] represents the area of a triangle whereas D-metric, represents the perimeter of the triangle. After introducing D-metric Dhage [8]-[9] developed some typologies in this structure. He also defined the concept of open balls and claimed that it forms a basis for some topology. Dhage presented topological structure in such spaces in his papers [7]-[9]. Dhage [8] proved some results on completeness and compactness of D-metric space to find Cantor's Intersection Theorem. Unfortunately most of the claims by Dhage are not remained true. As a result, it seems to be reasonable to revise most of the results of Dhage. R. Asim et al. [1] defined a modified open ball in this setting and consequently several authors modified various results of Dhage. In this paper, we take the concept of open balls defined by R. Asim et al. and show that it forms a basis for the D-metric topology on X. Also, we prove Cantor's intersection theorem and Baire's category theorem in D-metric space using it. Let us recall the definition of D-metric space.

Definition 1 (c.f [12]). *Let X be a non empty set. A function $D : X \times X \times X \to R^+$ is called D-metric on X if it satisfies the following properties,*

1. $D(x, y, z) \geq 0 \quad \forall x, y, z \in X$.
2. $D(x, y, z) = 0 \Leftrightarrow x = y = z$.
3. $D(x, y, z) = D(P\{x, y, z\})$ *where P is permutation on $\{x, y, z\}$.*
4. $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z) \quad \forall x, y, z, a \in X$.

Then the pair (X, D) is called a D-metric space.

2. Concept of Open Ball

First we present the notion of open ball in D-metric space in the line of B.C. Dhage [8] as follows.
Let $x_0 \in X$, $r > 0$ and an open ball centered at x_0 and radius r is

$$\hat{B}(x_0, r) = \bigcap_{y \in X} \{x, y \in X : D(x_0, x, y) < r\}.$$

Note that this definition is not an appropriate one and so most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect, that is there is a question mark on the validity of many results obtained by Dhage in these spaces, for detailed study one is referred to see the examples in [15]. Afterwards this definition was rectified by Naidu S.V.R [11] in the following way

$$B^*(x_0, r) = \{x \in X : D(x_0, x, x) < r\},$$

$$\hat{B}(x_0, r) = \{x_0\} \cup \{x \in X : \sup_{y \in X} D(x_0, x, y) < r\}.$$

It should be noted that the existence of $\sup_{y \in X} D(x_0, x, y) < r$ is not always possible. However $\sup_{y \in B^*(x_0, r)} D(x_0, x, y) < r$ always exists. Further, R. Asim et al. [1] believed that there is some typing error in defining $\hat{B}(x_0, r)$ and hence in [11] this concept may be refined in the following way:

Definition 2. *Let (X, D) be a D-metric space, $x_0 \in X$ and $r > 0$ then $\hat{B}(x_0, r)$ is called D-open or* **open ball** *if*

$$\hat{B}(x_0, r) = \{x_0\} \cup \{x \in X : \sup_{y \in B^*(x_0, r)} D(x_0, x, y) < r\}$$

where $B^*(x_0, r) = \{x \in X : D(x_0, x, x) < r\}$.

Dhage [[8], [9]] observed that a topology can be generated in X by taking the collection of all D-open balls as a sub-basis, which we call here the D-metric topology, to be denoted by τ. Thus (X, τ) is a D-metric topological space. Members of τ are called D- open sets and their complements, D-closed sets. But in [8], conclusion for $B(x_0, r)$ in theorems $3.1, 3.2, 3.3$, and 3.4 are false, for detailed see [11]. In this situation, we prove the following results by modified definition of open ball by R. Asim et al.

Remark. If $0 < r_1 < r_2$ then

(i) $B^*(x_0, r_1) \subset B^*(x_0, r_2)$.

(ii) $\hat{B}(x_0, r_1) \subset \hat{B}(x_0, r_2)$.

Definition 3. *A set U in a D-metric space is said to be open if it contains a ball of each of its points.*

Theorem 1. *Every ball $\hat{B}(x_0, r), x \in X, r > 0$ is an open set in X i.e. it contains a ball of each of its points.*

Proof. Let x_0 be an arbitrary point in X, $r > 0$. Consider the ball $\hat{B}(x_0, r)$ in X and let $x \in \hat{B}(x_0, r)$ then $\sup\limits_{a \in B^*(x_0,r)} D(x_0, x, a) = r_1 < r$ and $D(x_0, a, a) = r_2 < r$. Now we choose $0 < r_0 = max\{r_1, r_2\} < r$ then

$$\sup_{a \in B^*(x_0,r)} D(x_0, a, a) \leq r_0 < r_3 < r.$$

This implies $x \in \hat{B}(a, r_3) \subset \hat{B}(x_0, r), \quad r_3 > 0$. This proves that $\hat{B}(x_0, r)$ is an open set in X.

2.1. D-metric topology

In this section we discuss the topology on D-metric space X. We show that the collection $\beta = \{\hat{B}(x_0, r) : x \in X, r > 0\}$ of D-open balls induces a topology on X, called D-metric topology.

Theorem 2. *The collection β of all D-balls forms a basis for a topology τ on X.*

Proof. Let τ be a topology on X. To show that the collection β is a basis for τ it is enough to show that the collection β satisfies the following conditions:

(i) $X \subset (\cup_{x \in X} \hat{B}(x, r))$ and

(ii) if $x \in \hat{B}(x_1, r) \cap \hat{B}(x_2, r), r > 0$ for some $x_1, x_2 \in X$, is any point, then $\sup_{a \in B^*(x_1,r)} D(x_1, x, a) = s_1 < r$ and $\sup_{b \in B^*(x_2,r)} D(x_2, x, b) = s_2 < r$ for some $a \in B^*(x_1, r)$ and $b \in B^*(x_2, r)$. Therefore we choose $0 < s = max\{s_1, s_2\} < r$ then from remark (ii) we have $\hat{B}(x, r_0) \subset \hat{B}(x_1, r) \cap \hat{B}(x_2, r), s < r_0 < r$.

These complete the proof.

Thus the D-metric space X together with a topology τ generated by D-metric D is called a D-metric topological space and τ is called D-metric topology on X.

A topological space X is called D-metrizable if there exists a D-metric D on X that induces a topology on X. A D-metric space X is D-metrizable space together with the specific D-metric D that induces the topology of X.

A set U is τ-open in X in the D-metric topology τ induced by D-metric D if and only if for each $x \in U$, there is a $r > 0$ such that $\hat{B_D}(x, r) \subset U$. Similarly, a set V is called τ-closed if its compliment $X \setminus V$ is τ-open.

Lemma 3. *A subset U of (X, τ) is D-open if and only if for any $x \in U$ there are finite real numbers $r_1, r_2, ..., r_n > 0$ such that*

$$x \in \hat{B}(x, r_1) \cap ... \cap \hat{B}(x, r_n) \subset U.$$

Proof. Since each set of the form

$$\hat{B}(x, r_1) \cap ... \cap \hat{B}(x, r_n)$$

is D- open by definition, the sufficiency of the condition follows immediately.

Conversely let U be D-open and $x \in U$. Then there exists a finite number of D-balls $\hat{B}(x_i, r_i), \quad i = 1, 2, 3, ..., m$ (say) such that

$$x \in \bigcap_{i=1}^{m} \hat{B}(x_i, r_i) \subset U.$$

Since $x \in \hat{B}(x_i, r_i)$, so $D(x, x_i, y) = s_i < r_i$ where $y \in B^*(x_i, r_i)$. Choose $t_i < \frac{r_i - s_i}{2}$. Then $\hat{B}(x, t_i) \cap \hat{B}(x_i, t_i) \subset \hat{B}(x_i, r_i)$ and this is true for $i = 1, 2, 3, ..., m$. So

$$x \in \hat{B}(x, t_1) \cap \hat{B}(x_1, t_1) \cap \hat{B}(x, t_2) \cap \hat{B}(x_2, t_2) \cap \cdots \cap \hat{B}(x, t_m) \cap \hat{B}(x_m, t_m) \subset \bigcap_{i=1}^{m} \hat{B}(x_i, r_i) \subset U.$$

This proves the lemma.

Theorem 4. *Arbitrary union and finite intersection of open balls $\hat{B}(x, r), x \in X$ are open.*

Proof. Let $x \in \cup \hat{B}(x_i, r_i) \subset U$ then for some i, $x \in \hat{B}(x_i, r_i) \subset U$ hence U is open.
Next part of the theorem follows immediately from the Lemma 3.

Definition 4. *A set U in a D-metric space X, is said to be closed if its complement $X - U$ is τ open.*

Theorem 5. *Finite union and arbitrary intersection of closed balls in a D-metric space are closed.*

We omit the proof as it can be easily proved.

Definition 5. *$\bar{A}$ is called the D-closure of A if it is the intersection of all D-closed sets containing A.*

$\overline{B^*(x_0, r)} = \{x \in X | D(x_0, x, x) \leq r\}$ is the closure of $B^*(x_0, r)$ and
$\overline{\hat{B}(x_0, r)} = \{x \in X | \sup_{y \in B^*(x_0, r)} D(x_0, x, y) \leq r\}$ is the closure of $\hat{B}(x_0, r)$.

Remark. It is clear that $B^*(x_0, r) \subset \overline{B^*(x_0, r)}$ and $\hat{B}(x_0, r) \subset \overline{\hat{B}(x_0, r)}$.

Lemma 6. *If there exist a point $x \in \hat{B}(x_0, r)$ with $D(x_0, x, x) = r_1 < r$, then $\overline{\hat{B}(x_0, r_1)} \subset \hat{B}(x_0, r)$.*

Proof. Since $x \in \hat{B}(x_0, r)$ with $D(x_0, x, x) = r_1 < r$. Let

$$\begin{aligned} z \in \overline{\hat{B}(x_0, r_1)} &= \{z \in X | \sup_{x \in B^*(x_0, r_1)} D(x_0, z, x) \leq r_1\} \\ &\subseteq \{z \in X | \sup_{x \in B^*(x_0, r)} D(x_0, z, x) < r\} \\ &= \hat{B}(x_0, r). \end{aligned}$$

Definition 6. *$x \in (X, D)$, is called a D-limit point of $A \subset X$ if for any D-open set U containing x, $A \cap (U - \{x\}) \neq \phi$.*

Definition 7. *A sequence $\{x_n\}$ in a D-metric space (X, D) is said to be convergent (or D-convergent) if there exists an element x in X with the following property: given $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $D(x_m, x_n, x) < \varepsilon$ for all $m, n \geq n_0$.*

In such a case, it is said that $\{x_n\}$ converges to x and x is a limit point of $\{x_n\}$ and write $x_n \to x$.

Definition 8. *A sequence* $\{x_n\}$ *in a* D*-metric space* (X, D) *is said to be Cauchy (or* D*-Cauchy) if given* $\varepsilon > 0$ *there exists an* $n_0 \in \mathbb{N}$ *such that* $D(x_m, x_n, x_p) < \varepsilon$ *for all* $m, n, p \geq n_0$.

Definition 9. (X, D) *is said to complete if every Cauchy sequence in* X *is converges to a point in* X.

On the basis of definition of convergence of a sequence in (X, D) the following lemma is obtained.

Lemma 7. *A sequence* $\{x_n\}$ *is convergent to* x *in* (X, D) *if and only if for any* D*-open set* U *containing* x *there exists a positive integer* m *such that* $x_n \in U \ \forall \ n \geq m$.

Proof. Assume first the given condition. Let $x \in X$ and $\varepsilon > 0$. Since $\hat{B}(x, \varepsilon)$ is a D-open set containing x, there exists $m \in \mathbb{N}$ such that $x_n \in \hat{B}(x, \varepsilon) \ \ \forall \ \ n \geq m$ i.e. $D(x_n, x_n, x) < \varepsilon \ \forall \ n \geq m$ which shows that $D(x_n, x_n, x) \to 0$ as $n \to \infty$. Thus $\{x_n\}$ converges to x in (X, D).

Conversely let $\{x_n\}$ be convergent to x in (X, D). Let U be a D- open set with $x \in U$. From Lemma 3, we have

$$x \in \hat{B}(x, r_1) \cap \hat{B}(x_1, r_1) \cap \hat{B}(x, r_2) \cap \hat{B}(x_2, r_2) \cap ... \\ \cap \hat{B}(x, r_k) \cap \hat{B}(x_k, r_k) \subset U.$$

For some $x_1, x_2, ..., x_k \in X$ and $r_1, r_2, ..., r_k > 0$. Since $D(x_n, x_k, x) \to 0$ as $n \to \infty$, there exists $m_k \in \mathbb{N}$ such that $D(x_n, x, x_k) < r_k$ for all $n \geq m_k$ i.e. $x_n \in \hat{B}(x, r_k) \ \forall \ n \geq m_k$ and this is true for each $i = 1, 2, ..., k$. Taking $m = max\{m_1, ..., m_k\}$ we obtain

$$x_n \in \hat{B}(x, r_1) \cap \hat{B}(x_1, r_1) \cap \hat{B}(x, r_2) \cap \hat{B}(x_2, r_2) ... \\ \cap \hat{B}(x, r_k) \cap \hat{B}(x_k, r_k) \subset U, \ \forall \ n \geq m.$$

Thus the lemma is proved.

Note: It is known that in a metric space, a set A is closed if and only if every convergent sequence of points of A converges to a point of A.

Definition 10. $A \subset X$ *is said to be dense in* X *if* $\overline{A} = X$.

Definition 11. $A \subset X$ *is said to be no-where dense if* $int(\overline{A}) = \phi$ *where interior of a set* B *is defined to be the union of all* D*-open sets contained in* B.

3. Cantor's and Baire's Theorem in D-metric Spaces

Here we prove an analogue of Cantor's intersection theorem for complete D-metric spaces and use it to show that such a space cannot be expressed as a countable union of no-where dense sets under some general situations. For $A \subset X$, we define

$$\delta_c(A) = sup\{D(a,a,c) : \ a \in A\}$$

where $c \in X$.

The quantity $\delta_c(A)$ need not be considered as the diameter of A. However if (X,D) is bounded in the sense of Dhage and Rhoades [7], [8], and [13] (i.e. $sup\{D(a,b,c)\}; \ a,b,c \in X\} < \infty$) then for every $A \subset X, \ \delta_c(A)$ is finite $\forall \ c \in X$. The idea of $\delta_c(A)$ is helpful to prove the following theorems.

Theorem 8. *Suppose that (X,D) is a complete D-metric space. If $\{F_n\}$ is any decreasing sequence (i.e. $F_{n+1} \subset F_n \ \forall \ n \in \mathbb{N}$) of D-closed sets with $\delta_a(F_n) \to 0$ as $n \to \infty \ \ \forall \ a \in X$ then $\bigcap_{n=1}^{\infty} F_n$ is non-empty and contains at most one point.*

Proof. For each positive integer n, let x_n be a point of F_n. We show that $\{x_n\}$ is a Cauchy sequence in X. Since $\{F_n\}$ is decreasing, $x_m \in F_n \ \forall \ m \geq n$. Now for any $a \in X, \ \ m \geq n$,

$$D(x_m, x_n, a) \leq \delta_a(F_n) \to 0$$

as $n \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $x_n \to x$ (say) in X . We claim that $x \in \cap F_n$. We may assume that $x_k \neq x$ from some k onward, otherwise there is nothing to prove. Let $n \in \mathbb{N}$ be fixed. Let U be any D-open set containing x. By Lemma 7, there is $n_1 \in \mathbb{N}$ such that $x_k \in U \ \forall \ k \geq n_1$. Then $x_k \in [U - \{x\}] \cap F_n \ \forall \ k \geq max\{n, n_1\}$. This shows that $x \in \bar{F_n} = F_n$, since F_n is D-closed. As this is true for all $n \in \mathbb{N}, \ x \in \bigcap_{n=1}^{\infty} F_n$.

Finally, we prove that $\bigcap_{n=1}^{\infty} F_n$ contains at most one point. If possible let us suppose that it contains two distinct points x and y. Choose $z \in X, \ z \neq x \neq y$. From the definition of $\delta_z(F_n)$,

$$D(x,y,z) \leq \delta_z(F_n) \quad \ \forall \ n \in N.$$

Since $\delta_z(F_n) \to 0$ as $n \to \infty, \ D(x,y,z) = 0$ which is a contradiction. This proves the theorem.

To prove the converse of Theorem 8 the following lemma is necessary.

Lemma 9. *For any $A \subset X$ and $a \in X$*

$$\delta_a(A) = \delta_a(\bar{A}).$$

Proof. Since $A \subset \bar{A}$, it follows that $\delta_a(A) \leq \delta_a(\bar{A})$. To prove the converse inclusion, let $x, y \in \bar{A}$. If both x, y belong to A, then clearly $D(x, y, a) \leq \delta_a(A)$. So suppose first that one of them, say, $x \notin A$ but $y \in A$. Let $\varepsilon > 0$ be arbitrary. Since $x \in \bar{A}$. Now we consider D-open balls $y \in \hat{B}(x, \varepsilon)$ and $a \in \hat{B}(x, \varepsilon)$ containing x. Then $x \in \hat{B}(y, \varepsilon)$ and $x \in \hat{B}(a, \varepsilon)$ therefore, $x \in \hat{B}(y, \varepsilon) \cap \hat{B}(a, \varepsilon)$.

Since D-open balls form a basis for the topology defined on X there exists $\varepsilon_1 > 0$ such that $x \in \hat{B}(x, \varepsilon_1) \subset [\hat{B}(y, \varepsilon) \cap \hat{B}(a, \varepsilon)]$, where $0 < \varepsilon_1 < \varepsilon$.

Then as $x \in \bar{A}$ there exists z such that $z \in A \cap \hat{B}(x, \varepsilon_1)$ and this also implies $z \in A \cap \hat{B}(x, \varepsilon)$ as $z \in A \cap \hat{B}(x, \varepsilon_1) \subset A \cap \hat{B}(x, \varepsilon)$.

Therefore

$$\begin{aligned} D(x, y, a) &\leq D(x, y, z) + D(x, z, a) + D(z, y, a) \\ &\leq \delta_a(A) + 2\varepsilon. \end{aligned}$$

Since this is true for every $\varepsilon > 0$, we conclude that

$$D(x, y, a) \leq \delta_a(A) \quad for \quad y \in A \quad and \quad x \in \bar{A}.$$

Finally, if $x, y \in \bar{A} - A$ then repeating the same argument we can show that in this case also $D(x, y, a) \leq \delta_a(A)$. Hence

$$\delta_a(\bar{A}) = sup\{D(x, y, a);\ x, y \in \bar{A}\} \leq \delta_a(A)$$

and so $\delta_a(A) = \delta_a(\bar{A})$. This proves the lemma.

The converse of Theorem 8 is contained in the following theorem.

Theorem 10. *If in a D-metric space (X, D), for any decreasing sequence of D-closed sets $\{F_n\}$ with $\delta_a(F_n) \to 0$ as $n \to \infty$ $\forall$ $a \in X$, $\bigcap_{n=1}^{\infty} F_n$ consists of a single point then (X, D) is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Let $F_n = \{x_n, x_{n+1}, ...\}$ for any $n \in \mathbb{N}$. Then $F_n \supset F_{n+1}$ and so $\overline{F}_n \supset \overline{F}_{n+1}$ $\forall$ $n \in \mathbb{N}$. So $\{\overline{F}_n\}$ is a decreasing sequence of D-closed sets. For $a \in X$ and $\varepsilon > 0$ arbitrary, there is $n_1 \in \mathbb{N}$ such that

$$D(x_m, x_n, a) \quad < \quad \varepsilon \quad \forall\, m, n \geq n_1.$$

This shows that $\delta_a(F_{n_1}) \leq \varepsilon$ and so by Lemma 9 $\delta_a(\overline{F}_{n_1}) \leq \varepsilon$. Since $\{\overline{F}_n\}$ is decreasing, for $n \geq n_1$, $\delta_a(\overline{F}_n) \leq \delta_a(\overline{F}_{n_1}) \leq \varepsilon$. Therefore $\delta_a(\overline{F}_n) \to 0$ as $n \to \infty$.

Hence by the given condition, $\bigcap_{n=1}^{\infty} \overline{F}_n = \{x_0\}$, say. This gives that for any $a \in X$, $D(x_n, x_0, a) \leq \delta_a(\overline{F}_n) \to 0$ as $n \to \infty$ which implies $x_n \to x_0$ in X and this proves the theorem.

Combining Theorem 8 and Theorem 10, we obtain the analogue of Cantor's intersection theorem in D-metric spaces.

Theorem 11. *A D-metric space (X, D) is complete if and only if for any decreasing sequence of D- closed sets $\{F_n\}$ with $\delta_a(F_n) \to 0$ as $n \to \infty\ \forall\ a \in X$, $\bigcap_{n=1}^{\infty} F_n$ consists of a single point.*

The following lemma will be required for the next theorem.

Lemma 12. *For any $x_0 \in X$ and $r > 0$,*

$$\hat{C}(x_0, r) = \{x_0\} \cup \{x \in X; \sup_{y \in C^*(x_0,r)} D(x_0, x, y) \leq r\}$$

where $C^*(x_0, r) = \{x \in X : D(x_0, x, x) \leq r\}$ then $\hat{C}(x_0, r)$ is said to be a D-closed ball or D-closed set.

Proof. We will show that no point outside $\hat{C}(x_0, r)$ is a D-limit point of $\hat{C}(x_0, r)$. Let $d \notin \hat{C}(x_0, r)$. Then $D(x_0, y, d) > r$. If possible, let d be a D-limit point of $\hat{C}(x_0, r)$. Let $\varepsilon > 0$ be given. Since $\hat{B}(x_0, \epsilon) \cap \hat{B}(d, \varepsilon)$ is a D-open set containing d, there exists $e \in \hat{C}(x_0, r) \cap [[\hat{B}(x_0, \varepsilon) \cap \hat{B}(d, \varepsilon)] - \{d\}]$. Then

$$\begin{aligned} D(x_0, y, d) &\leq D(x_0, y, e) + D(y, e, d) + D(x_0, e, d) \\ &< r + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $D(x_0, y, d) \leq r$ which is a contradiction. Thus d cannot be a D-limit point of $\hat{C}(x_0, r)$. Hence $\hat{C}(x_0, r)$ contains all its D-limit points and so $\hat{C}(x_0, r)$ is D-closed. This proves the lemma.

In the next theorem, we prove an analogue of Baire's Category theorem for D-metric spaces.

Theorem 13. *A complete D-metric space (X, D) satisfying the condition (i) for every pair of points $x, y \in X$, there exists a sequence of D-closed balls $\{B_n\}$ with center at x and with $\delta_a(B_n) \to 0$ as $n \to \infty\ \ \forall\ a \in X$, cannot be written as a countable union of no-where dense sets.*

Proof. If possible, assume that

$$X = \bigcup_{n \in N} X_n = \bigcup_{n \in N} \overline{X}_n$$

where each X_n is no-where dense i.e. $\overline{X}_n$ does not contain any non-empty D- open set. Let U be any D- open set. Since X_1 is no-where dense, $\overline{X}_1$ cannot contain U. So there exists $x_1 \in U$ such that $x_1 \notin \overline{X}_1$. Since $U - \overline{X}_1$ is D- open and $x_1 \in U - \overline{X}_1$, by Lemma 3 there exists some real $r_1, r_2, ..., r_n$ all positive such that

$x_1 \in \hat{B}(x_1, r_1) \cap \hat{B}(x_1, r_2) \cap ... \cap \hat{B}(x_1, r_n)) = V_1$ (say) $\subset U - \overline{X}_1$. Without any loss of generality, because of the condition (i), we can choose $\hat{B}(x_1, r_1)$ such that $\delta_a(\hat{B}(x_1, r_1)) < 1 \; \forall \; a \in X$. Then $\delta_a(V_1) < 1 \; \forall \; a \in X$. Choose

$$U_1 = \hat{B}(x_1, r_1/2) \cap ... \cap \hat{B}(x_1, r_n/2).$$

Then by Lemma 12

$$\overline{U}_1 \subset \hat{C}(x_1, r_1/2) \cap ... \cap \hat{C}(x_1, r_n/2) \subset V_1 \quad \subset U - \overline{X}_1$$

and $\delta_a(\overline{U}_1) \leq \delta_a(V_1) < 1 \; \forall \; a \in X$. Again since U_1 is D- open and X_2 is no-where dense, $U_1 - \overline{X}_2 \neq \phi$. So there exists $x_2 \in U_1 - \overline{X}_2$. Proceeding as above we can find a D-open set U_2 such that

$$x_2 \in U_2 \subset \overline{U}_2 \subset U_1 - \overline{X}_2$$

and $\delta_a(\overline{U}_2) < 1/2 \; \forall \; a \in X$.

Continuing in this way we obtain a sequence of D- closed sets $\{\overline{U}_n\}$ such that $\overline{U}_{n+1} \subset \overline{U}_n \;\; \forall \, n \in N, \;\; \delta_a(\overline{U}_n) < 1/n \;\; \forall \, a \in X$ i.e. $\delta_a(\overline{U}_n) \to 0$ as $n \to \infty, \;\; \forall \;\; a \in X$. By Theorem 8, $\bigcap_{n=1}^{\infty} \overline{U}_n$ is non-empty and contains at most one point. Let

$$\bigcap_{n=1}^{\infty} \overline{U}_n = \{x_0\}.$$

Since $\overline{U}_n \cap \overline{X}_n = \phi \; \forall \; n \in N, \;\; x_0 \notin \bigcap_{n=1}^{\infty} \overline{X}_n$ which is a contradiction. This proves the theorem.

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