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COEFFICIENT BOUNDS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS

C. Selvaraj, T.R.K. Kumar, G. Thirupathi

Abstract. In the present investigation, we introduce two new subclasses $ST_\Sigma(b,\phi)$ and $CV_\Sigma(b,\phi)$ of bi-univalent functions defined in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. Besides, we find upper bounds for the second and third coefficients for functions in these new subclasses.

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1. Introduction, definitions and preliminaries

We let $\mathcal{A}$ to denote the class of functions analytic in $\mathbb{U}$ and having the power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also we let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. The Koebe one-quarter theorem [5] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$,$(z \in \mathbb{U})$ and

$$f\left(f^{-1}(w)\right) = w, \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

The coefficient estimate problem for the class $\mathcal{S}$, known as the Bieberbach conjecture, is settled by de-Branges [3], who proved that for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the class $\mathcal{S}$, $|a_n| \le n$, for $n = 2, 3, \cdots$, with equality only for the rotations of the Koebe function

$$K_0(z) = \frac{z}{(1-z)^2}.$$

Lewin [7]investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \le \sqrt{2}$.

An analytic function f is subordinate to an analytic function g,written $f(z) \prec g(z)$, provided there is a schwarz function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [8], unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk U, $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, (B_1 > 0). \tag{3}$$

Definition 1. *Let b be a non-zero complex number. A function $f(z)$ given by (1) is said to be in the class $ST_\Sigma(b, \phi)$ if the following conditions are satisfied:*

$$f \in \Sigma \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z), \quad z \in \mathbb{U} \tag{4}$$

$$\text{and} \quad 1 + \frac{1}{b}\left(\frac{wg'(w)}{g(w)} - 1\right) \prec \phi(w), \quad w \in \mathbb{U}, \tag{5}$$

where the function g is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

Definition 2. *Let b be a non-zero complex number. A function $f(z)$ given by (1) is said to be in the class $CV_\Sigma(b, \phi)$ if the following conditions are satisfied:*

$$f \in \Sigma \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) \prec \phi(z), z \in \mathbb{U} \tag{6}$$

$$\textit{and} \quad 1+\frac{1}{b}\left(\frac{wg''(w)}{g'(w)}\right) \prec \phi\left(w\right), w \in \mathbb{U}, \tag{7}$$

where the function g is given by

$$g\left(w\right)=f^{-1}\left(w\right)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots.$$

2. Coefficient estimates

Our first result provides estimates for the coefficients a_2, a_3 for functions belonging to the class $ST_\Sigma\left(b,\phi\right)$.

Theorem 1. *If $f \in ST_\Sigma\left(b,\phi\right)$, then*

$$|a_2| \le \frac{|b|\, B_1\sqrt{B_1}}{\sqrt{\left|B_1^2b+B_1-B_2\right|}} \quad \textit{and} \quad |a_3| \le \left(B_1+|B_2-B_1|\right)|b|. \tag{8}$$

Proof. Since $f \in ST_\Sigma\left(b,\phi\right)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0)=0=s(0)$, such that

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)=\phi\left(r(z)\right) \quad \text{and} \quad 1+\frac{1}{b}\left(\frac{wg'(w)}{g(w)}-1\right)=\phi\left(s(z)\right). \tag{9}$$

Define the functions p and q by

$$p\left(z\right)=\frac{1+r(z)}{1-r(z)}=1+p_1z+p_2z^2+\cdots \quad \text{and} \quad q\left(z\right)=\frac{1+s(z)}{1-s(z)}=1+q_1z+q_2z^2+\cdots. \tag{10}$$

Or equivalently,

$$r\left(z\right)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3+\frac{p_1}{2}\left(\frac{p_1^2}{2}-p_2\right)-\frac{p_1p_2}{2}\right)z^3+\cdots\right) \tag{11}$$

and

$$s\left(z\right)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left(q_1z+\left(q_2-\frac{q_1^2}{2}\right)z^2+\left(q_3+\frac{q_1}{2}\left(\frac{q_1^2}{2}-q_2\right)-\frac{q_1q_2}{2}\right)z^3+\cdots\right). \tag{12}$$

It is clear that p and q are analytic in $\mathbb{U}$ and $p(0)=1=q(0)$. Also p and q have positive real part in $\mathbb{U}$ and hence $|p_i| \le 2$ and $|q_i| \le 2$. In the view of (9),(11)and (12), clearly,

$$1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)=\phi\left(\frac{p(z)-1}{p(z)+1}\right) \quad \text{and} \quad 1+\frac{1}{b}\left(\frac{wg'(w)}{g(w)}-1\right)=\phi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{13}$$

Using (11) and (12) together with (3), one can easily verify that

$$\phi\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{B_1p_1}{2}z+\left(\frac{B_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2\right)z^2+\cdots \tag{14}$$

and

$$\phi\left(\frac{q(w)-1}{q(w)+1}\right)=1+\frac{B_1q_1}{2}w+\left(\frac{B_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{B_2q_1^2}{4}\right)w^2+\cdots. \tag{15}$$

Since $f\in\Sigma$ has the Maclaurin series given by (1), computation shows that its inverse $g=f^{-1}$ has the expansion given by (2). It follows from (13),(14) and (15) that

$$a_2=\frac{1}{2}B_1bp_1, \tag{16}$$

$$2a_3-a_2^2=\frac{1}{2}B_1b\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}B_2p_1^2b \tag{17}$$

and

$$a_2^3-3a_2a_3+3a_4=\frac{B_1b}{2}\left(2p_3+p_1\left(\frac{p_1^2}{2}-p_2\right)-p_1p_2\right)+\frac{B_2p_1b}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{B_3bp_1^3}{8}. \tag{18}$$

And

$$-a_2=\frac{1}{2}B_1bq_1, \tag{19}$$

$$3a_2^2-2a_3=\frac{1}{2}B_1b\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2b, \tag{20}$$

$$5a_2^3-15a_2^2+12a_2a_3-3a_4=\frac{B_1b}{2}\left(2q_3+q_1\left(\frac{q_1^2}{2}-q_2\right)-q_1q_2\right)+\frac{B_2q_1b}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{B_3bq_1^3}{8}. \tag{21}$$

From (16) and (19) , it follows that

$$p_1=-q_1, \tag{22}$$

and from (17)

$$a_3=\frac{a_2^2}{2}+\frac{1}{4}B_1bp_2-\frac{1}{8}B_1bp_1^2+\frac{1}{8}B_2p_1^2b. \tag{23}$$

Now (17),(20) and (23) gives

$$a_2^2 = \frac{B_1^3 b^2 (p_2 + q_2)}{4\left(B_1^2 b + B_1 - B_2\right)}. \tag{24}$$

Using the fact that $|p_2| \leq 2$ and $|q_2| \leq 2$ gives the desired estimate on $|a_2|$,

$$|a_2| \leq \frac{|b|\, B_1 \sqrt{B_1}}{\sqrt{\left|B_1^2 b + B_1 - B_2\right|}}.$$

From (17)-(22), gives

$$a_3 = \frac{\left(\frac{B_1 b}{2}\right)(3p_2 + q_2) + b p_1^2 (B_2 - B_1)}{4}.$$

Using the inequalities $|p_1| \leq 2$, $|p_2| \leq 2$ and $|q_2| \leq 2$ for functions with positive real part yields

$$|a_3| \leq (B_1 + |B_2 - B_1|)\,|b|\,.$$

For a choice of $\phi(z) = \dfrac{1 + Az}{1 + Bz}$, $-1 \leq B < A \leq 1$, we have the following corollary.

Corollary 2. *Let* $-1 \leq B < A \leq 1$. *If* $f \in ST_\Sigma\left(b, \frac{1+Az}{1+Bz}\right)$, *then*

$$|a_2| \leq \frac{|b|\,(A - B)}{\sqrt{|(A - B)\, b + (1 + B)|}}$$

and

$$|a_3| \leq |A - B|\,(1 + |1 + B|)\,|b|$$

.

If we let $\phi(z) = \left(\dfrac{1 + z}{1 - z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots$, $0 \leq \alpha < 1$, in the above theorem, we get the following corollary.

Corollary 3. *Let* $0 < \alpha \leq 1$. *If* $f \in ST_\Sigma(b, \alpha)$, *then*

$$|a_2| \leq \frac{2\alpha\,|b|}{\sqrt{|2\alpha b + (1 - \alpha)|}}$$

and

$$|a_3| \leq 2\alpha\,(1 + |\alpha - 1|)\,|b|\,.$$

Remark 1. *It is interesting to note that several well known and (presumably) new results can be obtained by specializing the function $\phi(z)$. For details see [1], [2].*

Analogous to the coefficient estimates obtained for the class $ST_{\Sigma}(b,\phi)$, we now proceed to obtain the coefficient estimates of the class $CV_{\Sigma}(b,\phi)$.

Theorem 4. *If $CV_{\Sigma}(b,\phi)$, then*

$$|a_2| \leq \frac{|b|\, B_1\sqrt{B_1}}{\sqrt{2\left|B_1^2 b + 2\left(B_1 - B_2\right)\right|}} \quad \textit{and} \quad |a_3| \leq \frac{\left(B_1 + |B_2 - B_1|\right)|b|}{2}. \tag{25}$$

Proof. Since $f \in CV_{\Sigma}(b,\phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, satisfying

$$1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) = \phi\left(r(z)\right) \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{wg''(w)}{g'(w)}\right) = \phi\left(s(w)\right). \tag{26}$$

Let p and q be defined as in (10), then it is clear from(26),(11)and (12)that

$$1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) = \phi\left(\frac{p(z)-1}{p(z)+1}\right) \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{wg''(w)}{g'(w)}\right) = \phi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{27}$$

It follows from (27),(15) and (16),

$$2a_2 = \frac{1}{2}B_1 b p_1, \tag{28}$$

$$6a_3 - 4a_2^2 = \frac{1}{2}B_1 b\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2 b, \tag{29}$$

$$-2a_2 = \frac{1}{2}B_1 b q_1, \tag{30}$$

and

$$8a_2^2 - 6a_3 = \frac{1}{2}B_1 b\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2 b. \tag{31}$$

The equations (28) and (30) yield

$$p_1 = -q_1 \tag{32}$$

and from (29) and(31), we get

$$a_2^2 = \frac{B_1^3 b^2\left(p_2 + q_2\right)}{8\left(B_1^2 b + 2\left(B_1 - B_2\right)\right)} \tag{33}$$

which yields the desired estimate on $|a_2|$ as described in 25.

Similarly, it can be obtained from (29)-(31),

$$a_3 = \frac{\left(\frac{B_1 b}{2}\right)(2p_2+q_2) + \left(\frac{3bp_1^2}{4}\right)(B_2-B_1)}{6}$$

which yields the estimate (25).

Corollary 5. *Let $-1 \le B < A \le 1$. If $f \in CV_\Sigma\left(b, \frac{1+Az}{1+Bz}\right)$, then*

$$|a_2| \le \frac{|b|(A-B)}{\sqrt{2\left|(A-B)b + 2(1+B)\right|}}$$

and

$$|a_3| \le \frac{|A-B|\,(1+|1+B|)\,|b|}{2}.$$

Corollary 6. *Let $0 < \alpha \le 1$. If $f \in CV_\Sigma(b,\alpha)$, then*

$$|a_2| \le \frac{|b|\,\alpha}{\sqrt{|\alpha b + (1-\alpha)|}}$$

and

$$|a_3| \le \alpha\,(1+|\alpha-1|)\,|b|\,.$$

3. Coefficients bounds for the function class $M_\Sigma(\alpha,\phi)$

Theorem 7. *Let f given by (1) be in the class $M_\Sigma(\alpha,\phi)$, then*

$$|a_2| \le \frac{|b|\,B_1\sqrt{B_1}}{\sqrt{(1+\alpha)\left|B_1^2 b + (1+\alpha)(B_1-B_2)\right|}} \quad \text{and} \quad |a_3| \le \frac{(B_1+|B_2-B_1|)\,|b|}{(1+\alpha)}. \tag{34}$$

Proof. If $f \in M_\Sigma(\alpha,\phi)$, then there are analytic functions $r,s:\mathbb{U}\to\mathbb{U}$, with $r(0) = 0 = s(0)$ such that

$$(1-\alpha)\left(1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right) + \alpha\left(1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\right) = \phi(r(z)), \tag{35}$$

$$(1-\alpha)\left(1+\frac{1}{b}\left(\frac{wg'(w)}{g(w)}-1\right)\right) + \alpha\left(1+\frac{1}{b}\frac{wg''(w)}{g'(w)}\right) = \phi(s(w)). \tag{36}$$

From (14),(15),(35)and(36), it follows that

$$(1+\alpha)\,a_2 = \frac{1}{2}B_1 b p_1, \tag{37}$$

$$(2+4\alpha)\,a_3 - (1+3\alpha)\,a_2^2 = \frac{1}{2}B_1 b\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2 b, \tag{38}$$

$$-(1+\alpha)\,a_2 = \frac{1}{2}B_1 b q_1 \tag{39}$$

and

$$(3+5\alpha)\,a_2^2 - (2+4\alpha)\,a_3 = \frac{1}{2}B_1 b\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2 b. \tag{40}$$

From (37),(39),

$$p_1 = -q_1. \tag{41}$$

The equations (38),(40)and(41), gives

$$a_2^2 = \frac{B_1^3 b^2\,(p_2+q_2)}{4\,(1+\alpha)\left(B_1^2 b + (1+\alpha)\,(B_1 - B_2)\right)} \tag{42}$$

which yields the desired estimation of $|a_2|$ in (34).
Subtracting (38) from (40) , using (41), gives

$$a_3 = \frac{\frac{B_1 b}{2}\left((3+5\alpha)\,p_2 + (1+3\alpha)\right) + b p_1^2\,(B_2 - B_1)}{4\,(1+2\alpha)\,(1+\alpha)}. \tag{43}$$

Using the familiar inequalities $|p_i| \leq 2$ and$|q_i| \leq 2$, (43) gives

$$|a_3| \leq \frac{(B_1 + |B_2 - B_1|)\,|b|}{(1+\alpha)}. \tag{44}$$

Remark 2. *If we let $b = 1$, Theorem 3.1 reduce to the result of R.M.Ali et.al [2].*

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Chellapan Selvaraj
Presidency College
Chennai-600 005,
Tamilnadu, India.
email: *pamc9439@yahoo.co.in.* T.R.K.Kumar

R.M.K.Engineering College
R.S.M.Nagar, Kavaraipettai-601 206,
Tamilnadu, India.
email: *kumartrk@yahoo.com.* T.R.K.Kumar

R.M.K.Engineering College
R.S.M.Nagar, Kavaraipettai-601 206,
Tamilnadu, India.
email: *gt_venkat79@yahoo.com.*