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SOME SUBORDINATION THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING A LINEAR OPERATOR

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Abstract. By using the subordination theorem for analytic functions we derive interesting subordination results for certain class of analytic functions defined by new linear operator.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disk $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in $\mathbf{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbf{U})$, if there exists a Schwarz function $w(z)$ in $\mathbf{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbf{U})$, such that $f(z) = g(w(z))$, $(z \in \mathbf{U})$. In particular, if $g(z)$ is univalent in $\mathbf{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$ (see [16] and [17]).

For the functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

Let CV and ST be the subclasses of $\mathcal{A}$ which are starlike and convex functions, respectively. A function $f(z) \in \mathcal{A}$ is said to be in the class of uniformly starlike functions of order γ and type β, denoted by $SP(\beta, \gamma)$ if

$$\Re\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{4}$$

where $\beta \geq 0, -1 \leq \gamma < 1, \beta + \gamma \geq 0$. Similarly, if $f(z) \in \mathcal{A}$ satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \tag{5}$$

where $\beta \geq 0, -1 \leq \gamma < 1, \beta + \gamma \geq 0$, then $f(z)$ is said to be in the class of uniformly convex functions of order γ and type β, and is denoted by $UCV(\beta, \gamma)$. The classes $SP(\beta, \gamma)$ and $UCV(\beta, \gamma)$ were studied by Bharti et al. [8].

For functions $f, g \in \mathcal{A}$, we define the linear operator $D_\lambda^n : \mathcal{A} \to \mathcal{A}$ $(\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\})$ by:

$$D_\lambda^0(f * g)(z) = (f * g)(z)\,,$$

$$D_\lambda^1(f * g)(z) = D_\lambda(f * g)(z) = (1 - \lambda)(f * g)(z) + \lambda z\left((f * g)(z)\right)',$$

and (in general)

$$D_\lambda^n(f * g)(z) = D_\lambda(D_\lambda^{n-1}(f * g)(z)) \quad (\lambda \geq 0; n \in \mathbb{N})\,. \tag{6}$$

If f and g are given by (1) and (2), respectively, then from (6), we see that

$$D_\lambda^n(f * g)(z) = z + \sum_{k=2}^{\infty}[1 + \lambda(k-1)]^n a_k b_k z^k \quad (\lambda \geq 0; n \in \mathbb{N}_0)\,. \tag{7}$$

From (7), we can easily deduce that

$$\lambda z\left(D_\lambda^n(f * g)(z)\right)' = D_\lambda^{n+1}(f * g)(z) - (1 - \lambda)D_\lambda^n(f * g)(z)\ (\lambda > 0). \tag{8}$$

The operator $D_\lambda^n(f * g)(z)$ was introduced by Aouf and Seoudy [5]. We observe that the linear operator $D_\lambda^n(f * g)(z)$ reduces to several interesting many other linear operators considered earlier for different choices of n, λ and the function $g(z)$:

(i) For $b_k = 1$ (or $g(z) = \dfrac{z}{1-z}$), we have $D_\lambda^n(f * g)(z) = D_\lambda^n f(z)$, where D_λ^n is the generalized Sălăgean operator (or Al-Oboudi operator [1]) which yield Sălăgean operator D^n for $\lambda = 1$ introduced and studied by Sălăgean [22];

(*ii*) For $n = 0$ and

$$b_k = \Gamma_k = \frac{(a_1)_{k-1}...(a_l)_{k-1}}{(b_1)_{k-1}...(b_m)_{k-1}(1)_{k-1}} \tag{9}$$

$$\left(a_i \in \mathbb{C}; i = 1,..,l; b_j \in \mathbb{C}\backslash\mathbb{Z}_0^- = \{0,-1,..\}; j = 1,..,m; l \leq m+1; l,m \in \mathbb{N}_0\right),$$

where

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & (k = 0; x \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ x(x+1)...(x+k-1) & (k \in \mathbb{N}; x \in \mathbb{C}), \end{cases}$$

we have $D_\lambda^0(f*g)(z) = (f*g)(z) = H_{l,m}(a_1; b_1) f(z)$, where the operator $H_{l,m}(a_1; b_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [10] (see also [11] and [12]). The operator $H_{l,m}(a_1; b_1)$, contains in turn many interesting operators such as, Hohlov linear operator (see [13]), the Carlson-Shaffer linear operator (see [9] and [21]), the Ruscheweyh derivative operator (see [20]), the Bernardi-Libera-Livingston operator (see [7], [14] and [15]) and Owa-Srivastava fractional derivative operator (see [18]);

(*iii*) For $g(z)$ of the form (9), the operator $D_\lambda^n(f*g)(z) = D_\lambda^n(a_1, b_1)f(z)$, introduced and studied by Selvaraj and Karthikeyan [23];

(*iv*) For

$$b_k = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right]^n \quad (\alpha \neq 2,3,4,...),$$

we have $D_\lambda^n(f*g)(z) = D_\lambda^{n,\alpha}f(z)$, where $D_\lambda^{n,\alpha}f(z)$ is a linear operator which was introduced and studied by Al-Oboudi and Al-Amoudi ([2] and [3], see also [4]);

(*v*) For

$$b_k = \left[\frac{(a)_{k-1}}{(c)_{k-1}}\right]^n \quad \left(a, c \in \mathbb{R}^+\right),$$

we note that $D_\lambda^n(f*g)(z) = I_{a,c,\lambda}^n f(z)$, where $I_{a,c,\lambda}^n f(z)$ is a linear multiplier operator which introduced by Prajapat and Riana [19];

(*vi*) For $b_k = [\Gamma_k]^n$, where Γ_k is given by (1.9), we obtain the linear operator $D_\lambda^n(f*g)(z) = L_{\lambda,l,m}^n(a_1; b_1) f(z)$, where $L_{\lambda,l,m}^n(a_1; b_1)$ is defined by Srivastava et al. [24]. The operator $L_{\lambda,l,m}^n(a_1; b_1)$ contains Al-Oboudi and Al-Amoudi operator [2, 3] and Prajapat and Riana operator [19].

Let $SP_\lambda^n(f, g; \gamma, \beta)$ be the class of functions $f, g \in \mathcal{A}$ satisfying the following condition:

$$\Re\left\{\frac{z(D_\lambda^n(f*g)(z))'}{D_\lambda^n(f*g)(z)} - \gamma\right\} > \beta\left|\frac{z(D_\lambda^n(f*g)(z))'}{D_\lambda^n(f*g)(z)} - 1\right| \quad (z \in \mathbf{U}), \tag{10}$$

where $-1 \leq \gamma < 1$, $\beta \geq 0$, $\beta+\gamma \geq 0, \lambda \geq 0$ and $n \in \mathbb{N}_0$.

Let $UCV_\lambda^n(f,g;\gamma,\beta)$ be the class of function $f,g \in A$ satisfying the following condition:

$$\Re\left\{1+\frac{z\left(D_\lambda^n(f*g)(z)\right)''}{\left(D_\lambda^n(f*g)(z)\right)'}-\gamma\right\} > \beta\left|\frac{z\left(D_\lambda^n(f*g)(z)\right)''}{\left(D_\lambda^n(f*g)(z)\right)'}\right| \quad (z \in \mathbf{U}), \tag{11}$$

where $-1 \leq \gamma < 1$, $\beta \geq 0$, $\beta+\gamma \geq 0, \lambda \geq 0$ and $n \in \mathbb{N}_0$.

From (10) and (11), we have

$$f(z) \in UCV_\lambda^n(f,g;\gamma,\beta) \Leftrightarrow zf'(z) \in SP_\lambda^n(f,g;\gamma,\beta). \tag{12}$$

Taking $b_k = [\Gamma_k]^n$, where Γ_k is given by *(9)*, we note that $SP_\lambda^n(f,g;\gamma,\beta) = SP_{\lambda,l,m}^n(a_1;b_1;\gamma,\beta)$ and $UCV_\lambda^n(f,g;\gamma,\beta) = UCV_{\lambda,l,m}^n(a_1;b_1;\gamma,\beta)$.

Definition 1. *[25] A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1) is analytic, univalent and convex in $\mathbf{U}$, we have*

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \qquad (z \in \mathbf{U}; a_1 = 1) . \tag{13}$$

2. Main Results

To state and prove our main results, we need the following lemma.

Lemma 1. *[25] The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left(1+2\sum_{k=1}^{\infty} c_k z^k\right) > 0 \qquad (z \in \mathbf{U}) . \tag{14}$$

Theorem 2. *A function $f(z) \in \mathcal{A}$ of the form (1) is in the class $SP_\lambda^n(f,g;\gamma,\beta)$ if*

$$\sum_{k=2}^{\infty} \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\lambda(k-1)\right]^n |b_k|\,|a_k| \leq 1-\gamma, \tag{15}$$

where $g(z)$ is given by (2), $-1 \leq \gamma < 1, \beta \geq 0, \lambda \geq 0$ and $n \in \mathbb{N}_0$.

Proof. It suffices to show that

$$\beta\left|\frac{z(D_\lambda^n(f*g)(z))'}{D_\lambda^n(f*g)(z)}-1\right|-\Re\left\{\frac{z(D_\lambda^n(f*g)(z))'}{D_\lambda^n(f*g)(z)}-1\right\}<1-\gamma \quad (z\in\mathbf{U}),$$

we have

$$\begin{aligned}
&\beta\left|\frac{z(D_\lambda^n(f*g)(z))'}{D_\lambda^n(f*g)(z)}-1\right|-\Re\left\{\frac{z(D_\lambda^n(f*g)(z))'}{D_\lambda^n(f*g)(z)}-1\right\}\\
\leq\;&(1+\beta)\left|\frac{z(D_\lambda^n(f*g)(z))'}{D_\lambda^n(f*g)(z)}-1\right|\\
\leq\;&\frac{(1+\beta)\sum\limits_{k=2}^{\infty}(k-1)\left[1+\lambda(k-1)\right]^n|b_k|\,|a_k|\,|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^n|b_k|\,|a_k|\,|z|^{k-1}}\\
<\;&\frac{(1+\beta)\sum\limits_{k=2}^{\infty}(k-1)\left[1+\lambda(k-1)\right]^n|b_k|\,|a_k|}{1-\sum\limits_{k=2}^{\infty}\left[1+\lambda(k-1)\right]^n|b_k|\,|a_k|}.
\end{aligned}$$

This last expression is bounded above by $(1-\gamma)$ if (14) is satisfied.

By virture of (12) and Theorem 2, we have

Corollary 3. *A function $f(z)\in\mathcal{A}$ of the form (1) is in the class $UCV_\lambda^n(f,g;\gamma,\beta)$ if*

$$\sum_{k=2}^{\infty}k\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\lambda(k-1)\right]^n|b_k|\,|a_k|\leq 1-\gamma,$$

where $g(z)$ is given by (2), $-1\leq\gamma<1,\ \beta\geq 0,\ \lambda\geq 0$ and $n\in\mathbb{N}_0$.

Let $SP_\lambda^{n*}(f,g;\gamma,\beta)$ and $UCV_\lambda^{n*}(f,g;\gamma,\beta)$ denote the classes of functions $f(z)\in\mathcal{A}$ of the form (1) whose coefficients satisfy the conditions (15) and (16), respectively. We note that $SP_\lambda^{n*}(f,g;\gamma,\beta)\subseteq SP_\lambda^{n}(f,g;\gamma,\beta)$ and $UCV_\lambda^{n*}(f,g;\gamma,\beta)\subseteq UCV_\lambda^{n}(f,g;\gamma,\beta)$.

Theorem 4. *Let the function $f(z)$ defined by (1) be in the class $SP_\lambda^{n*}(f,g;\gamma,\beta)$, where $g(z)$ is given by (2), $\beta\geq 0,\ -1\leq\gamma<1,\lambda\geq 0$ and $n\in\mathbb{N}_0$. Then*

$$\frac{(2+\beta-\gamma)(1+\lambda)^n|b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n|b_2|]}(f*h)(z)\prec h(z)\quad (z\in\mathbf{U};h\in CV) \qquad (16)$$

and

$$\Re(f(z)) > -\frac{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|}{(2+\beta-\gamma)(1+\lambda)^n |b_2|} \quad (z \in \mathbf{U}). \tag{17}$$

The constant $\dfrac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|]}$ is the best estimate.

Proof. Let $f(z) \in SP_{\lambda}^{n*}(f,g;\gamma,\beta)$ and suppose that $h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in CV$. Then we readily have

$$\frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|]} (f*h)(z)$$

$$= \frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|]} \left(z + \sum_{k=2}^{\infty} a_k c_k z^k \right). \tag{18}$$

Thus, by Definition 1, the assertion of our theorem will hold if the sequence

$$\left\{ \frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|]} a_k \right\}_{k=1}^{\infty} \tag{19}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality

$$\Re\left\{ 1 + \sum_{k=1}^{\infty} \frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|} a_k z^k \right\} > 0 \quad (z \in \mathbf{U}). \tag{20}$$

Now since

$$[k(1+\beta)-(\gamma+\beta)]\,[1+\lambda(k-1)]^n \quad (\beta \geq 0; -1 \leq \gamma < 1; \lambda > 0; n \in \mathbb{N}_0)$$

is an increasing function of k, we have

$$\Re\left\{ 1 + \sum_{k=1}^{\infty} \frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|} a_k z^k \right\}$$

$$= \Re\left\{ 1 + \frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|} z + \frac{\sum_{k=2}^{\infty} (2+\beta-\gamma)(1+\lambda)^n |b_2|\, a_k z^k}{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|} \right\}$$

$$\geq 1 - \frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|} r - \frac{\sum_{k=2}^{\infty} [k(1+\beta)-(\alpha+\beta)]\,[1+\lambda(k-1)]^n |b_k|\, |a_k|\, r^k}{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|}$$

$$> 1 - \frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|} r - \frac{1-\gamma}{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|} r$$

$$= 1 - r > 0 \quad (|z| = r < 1), \tag{21}$$

where we have used the assertion (15) of Theorem 2. Thus (20) holds true in $\mathbf{U}$. This proves the first assertion. The inequality (17) follows from (16) by taking

$$h(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in CV \ . \tag{22}$$

To prove the sharpness of the constant $\frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|]}$, we consider the function $f_0(z)$ defined by

$$f_0(z) = z - \frac{1-\gamma}{(2+\beta-\gamma)(1+\lambda)^n |b_2|} z^2 \quad (\beta \geq 0; -1 \leq \gamma < 1; \lambda > 0; n \in \mathbb{N}_0), \tag{23}$$

which is a member of the class $SP_\lambda^{n*}(f, g; \gamma, \beta)$. Then from the relation (16), we obtain

$$\frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|]} f_0(z) \prec \frac{z}{1-z} \ . \tag{24}$$

It can be easily verified that

$$\min_{|z|\leq 1} \Re \left(\frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|]} \right) = -\frac{1}{2}, \tag{25}$$

this shows that the constant $\frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n |b_2|]}$ is best possible, and the proof of Theorem 4 is completed.

Similarly from (12) and Theorem 4, we can prove the following theorem.

Theorem 5. *Let the function $f(z)$ defined by (1) be in the class $UCV_\lambda^{n*}(f, g; \gamma, \beta)$, where $g(z)$ is given by (2), $\beta \geq 0$, $-1 \leq \gamma < 1, \lambda \geq 0$ and $n \in \mathbb{N}_0$. Then*

$$\frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{1-\gamma+2(2+\beta-\gamma)(1+\lambda)^n |b_2|} (f * h)(z) \prec h(z) \quad (z \in \mathbf{U}; h \in CV) \tag{26}$$

and

$$\Re(f(z)) > -\frac{1-\gamma+2(2+\beta-\gamma)(1+\lambda)^n |b_2|}{2(2+\beta-\gamma)(1+\lambda)^n |b_2|} \quad (z \in \mathbf{U}). \tag{27}$$

The constant $\frac{(2+\beta-\gamma)(1+\lambda)^n |b_2|}{1-\gamma+2(2+\beta-\gamma)(1+\lambda)^n |b_2|}$ is the best estimate.

Remark 1. *(i) Taking $b_k = 1$ in Theorem 4, we obtain the result of Aouf et al. [6, Theorem 1];*

(ii) Taking

$$b_k = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right]^n \quad (\alpha \neq 2,3,4,...),$$

in Theorems 4 and 4, respectively, we obtain the results of Aouf and Mostafa [4, Theorems 2.4 and 2.8, respectively];

(iii) Taking

$$b_k = \left[\frac{(a)_{k-1}}{(c)_{k-1}}\right]^n \quad (a, c \in \mathbb{R}^+),$$

in Theorem 4, we obtain the result of Prajapat and Riana [19, Theorem 1].

Taking $b_k = [\Gamma_k]^n$, where Γ_k is given by (9), in Theorems 4 and 5, we obtain the following results for the classes $SP^n_{\lambda,l,m}(a_1; b_1; \gamma, \beta)$ and $UCV^{n*}_{\lambda,l,m}(a_1; b_1; \gamma, \beta)$, respectively.

Corollary 6. *Let the function $f(z)$ defined by (1) be in the class $SP^n_{\lambda,l,m}(a_1; b_1; \gamma, \beta)$, where $g(z)$ is given by (2), $\beta \geq 0$, $-1 \leq \gamma < 1, \lambda \geq 0$ and $n \in \mathbb{N}_0$. Then*

$$\frac{(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|]}(f*h)(z) \prec h(z) \quad (z \in \mathbf{U}; h \in CV)$$

and

$$\Re(f(z)) > -\frac{1-\gamma+(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|}{(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|} \quad (z \in \mathbf{U}).$$

The constant $\dfrac{(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|}{2[1-\gamma+(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|]}$ *is the best estimate.*

Corollary 7. *Let the function $f(z)$ defined by (1) be in the class $UCV^{n*}_{\lambda,l,m}(a_1; b_1; \gamma, \beta)$, where $g(z)$ is given by (2), $\beta \geq 0$, $-1 \leq \gamma < 1, \lambda \geq 0$ and $n \in \mathbb{N}_0$. Then*

$$\frac{(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|}{1-\gamma+2(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|}(f*h)(z) \prec h(z) \quad (z \in \mathbf{U}; h \in CV)$$

and

$$\Re(f(z)) > -\frac{1-\gamma+2(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|}{2(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|} \quad (z \in \mathbf{U}).$$

The constant $\dfrac{(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|}{1-\gamma+2(2+\beta-\gamma)(1+\lambda)^n \left|[\Gamma_2]^n\right|}$ is the best estimate.

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