SUFFICIENT CONDITIONS FOR HAMILTONIANCITY OF CERTAIN SPECIAL GRAPHS

D.O. Ajayi and T.C. Adefokun

Abstract. In this article, Ore-type conditions for certain class of graphs, $G$, to be Hamiltonian are established. It involves partitioning vertex set $V(G)$ of $G$ into two subvertices, with specific conditions on the degrees of their of vertices such that for several distance-2 vertices $v, u \in V(G)$, $d(v) + d(u)$ can be much less than the order of $G$, particularly as $|V(G)| \to \infty$.

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1. Preliminaries

Here we present the existing results and some definitions needed in the work.

**Theorem 1.** (Dirac [1]): If $G$ is a simple graph with $n$ vertices, where $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then $G$ is Hamiltonian.

This result by Dirac was improved by Ore in the next result.

**Theorem 2.** (Ore [4]). Let $G$ be a simple graph with $n$ vertices and $u,v$ be distinct nonadjacent vertices of $G$ with $d(u) + d(v) \geq n$, then $G$ is Hamiltonian.

More recently, Li et. al.[3] presented a result that improved Ore’s result for certain graphs.

**Theorem 3.** Let $G$ be a 2-connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of vertices $u$ and $v$ with $d(u,v) = 2$, then $G$ is Hamiltonian unless $n$ is odd and $G \in L_n$

For the definition of $L_n$, see [3].
We define $[a, b]$ as the set of integers $\{a, a + 1, a + 2, ..., b\}$
2. Main Results

We begin with the next lemma.

**Lemma 4.** Let $G$ be a simple connected graph with $|V(G)| \leq \infty$. If $G$ is a 2-regular graph, then $G$ is a cycle.

**Proof.** Let $n \geq 3$ be a positive integer and $|V(G)| = n$. For $u_0, u_1 \in V(G)$, let $u_0u_1 \in E(G)$. Since $G$ is a simple and 2-regular, then there exists $u_2 \in V(G)$ such that $u_2 \neq u_0$, such that $u_1u_2 \in E(G)$. Since $G$ is connected and 2-regular, and with an iteration based on the last statement, there exist a path $P_n = v_0v_1v_2...v_{n-1}$ in $G$ and it consists of all the vertices in $G$. Now, for all $v_i \in V(P_n), i \neq 0, n - 1, d(v_i) = 2$. Now, since $G$ is 2-regular then $u_0u_i \notin E(G)$ for all $i \neq 1, n - 1$. Since $u_{n-2}u_{n-1} \in E(G)$, then $u_0u_{n-1} \in E(G)$ and thus, $G$ is a cycle.

Using the lemma, we obtain the main results:

**Theorem 5.** Let $G$ be a simple connected graph with $|V(G)| \geq 3$, and $|V(G)| \equiv 0 \pmod{3}$. Suppose $V(G)$ is partitioned into $V$ and $U$ with $|U| = \lfloor \frac{|V|}{2} \rfloor$ for each $u_i \in U$, $V \subseteq N_G(u_i)$. Suppose further that for $V = u_0, u_1, ..., u_{n+1}$, there exist a $E(V) = \{u_0u_1, u_2u_3, ..., u_{m-2}u_{m-1}\} \subseteq E(G)$, such that $|E(V)| = |U|$, then $G$ is Hamiltonian.

**Proof.** From the hypothesis, $d(u_i) \geq |V|$ for all $u_i \in U$ since $N_G(u_i) = |V|$. Therefore each $u_i \in U$ is incident to all $e_i \in E(V)$. Thus, suppose $U = \{u_0, u_1, ..., u_{n-1}\}$ and $V = \{v_0, v_1, ..., v_{m-1}\}$. For each $u_i \in U, i \in [1, n - 2]$, let $u_iv_{2i-1} \in E(G)$ and also $u_iv_{2(i+1)} \in E(G)$. Likewise, for some $u_0 \in U, u_0v_0, u_0v_2 \in E(G)$ and $u_{n-1}v_{m-3}, u_{n-1}v_{m-1} \in E(G)$. Thus each vertex on every member of $E(V)$ is incident to some vertex in $U$ and suppose every other edge in $G$ is deleted, the resultant graph say, $G'$, remains connected and for every $v \in V(G')$, $d(v) = 2$. Thus by Lemma 4, $G'$ is a spanning cycle of $G$ and hence, $G$ is Hamiltonian.

Since $|V(G)| \equiv 0 \pmod{3}$ in Theorem 5 above, it is easy to see that $|V(G)| - |U|$ is even. Thus, the vertices in $V$ can be paired into edges in $E(V)$. The next results take care of situations that are different.

**Theorem 6.** Let $G$ be a connected graph of order $|V(G)|$ with $|V(G)| \equiv 1 \pmod{3}$ and let $V(G)$ be partitioned into $U$ and $V$ with $|U| = \left\lfloor \frac{|V(G)|}{3} \right\rfloor$ with $V \subseteq N_G(u_i)$ for all $u_i \in U$. Suppose there exists a path $P_3 \subseteq G$ such that $V(P_3) \subseteq V$, and suppose $V'$ is defined as $V' = \{v_0, v_1, ..., v_{k-1}\} = V \setminus V(P_3)$. If for all $v_i \in V'$, there exists $E(V') = \{v_0v_1, v_2v_3...v_{k-2}v_{k-1}\} \subseteq E(G)$, then $G$ is Hamiltonian.
We should note that since for any positive integer \( p \), \( |V(G)| = 3p + 1 \) then, 
\[ |U| = \left\lfloor \frac{|V(G)|}{3} \right\rfloor = p. \] Therefore \( |V(G)| - |U| \) is odd. However, \( |V(P_3)| = 3 \) and since \( V' = V\backslash V(P_3) \), then \( k \) is even and thus members of \( V' \) can be paired.

Now we proceed to proof Theorem 6.

**Proof.** It is easy to see from the hypothesis that \( |V'| = |U| - 1 \). Now, suppose that path \( P_3 = v_1v_{i+1}v_{i+2} \), where \( \{v_{i+j}\}_{j=0}^{2} \subseteq V \) is a set of arbitrary vertices in \( V \). Obviously, since \( U \subseteq N_G(v_{i+1}) \), \( d(v_{i+1}) = 2 + |U| \). Let \( E(v_{i+1}) \) be the set of all edges associated with \( v_{i+1} \) and let \( E'(v_{i+1}) = E(v_{i+1}\backslash \{v_1v_{i+1}, v_{i+1}v_{i+2}\}) \). Now suppose we delete \( E'(v_{i+1}) \) then \( d(v_{i+1}) = 2 \). Thus, if there exists a spanning cycle \( C_{|V(G)|} \) in \( G\backslash E'(v_{i+1}) \), then \( P_3 \subseteq C_{|V(G)|} \). Thus, we 'shunt' \( P_3 \) into edge \( v_iv_{i+2} \) such that \( E(V') \cup v_iv_{i+2} = E(V'') \). Clearly, \( |E(V'')| = |U| \). Thus the claim follows from Lemma 4 and Theorem 5.

It should be noted, however, that there is an interesting relationship between the length of the path and the order of \( |U| \) in 6. This is expressed in the following corollary.

**Corollary 7.** Let \( G \) be as in 6. If \( |U| \) is reduced to \( |U| - r \) as \( r \rightarrow |U| - 2 \), and path \( P_3 \) extends to \( P_{3+r} \) also \( r \rightarrow |U| - 2 \), then \( G \) is Hamiltonian. Furthermore, if \( |U| = 2 \), then \( G \) is a cycle.

In the next theorem, we consider the second situation where \( |V(G)| \equiv 2 \mod 3 \).

**Theorem 8.** Let \( G \) be a connected graph of order \( |V(G)| \) with \( |V(G)| \equiv 2 \mod 3 \) and let \( V(G) \) be partitioned into \( U \) and \( V \) with \( |U| = \left\lfloor \frac{|V(G)|}{3} \right\rfloor \) with \( V \subseteq N_G(u_i) \) for all \( u_i \in U \). Suppose that, except for some \( v_k \in V \), for all \( v_i \in V' = \{v_0, v_1, \ldots, v_m-1\} = V\backslash v_k \), there exist \( E(V') = \{v_0v_1, v_2v_3, \ldots, v_{m-2}v_{m-1}\} \subset E(G) \). Then \( G \) is Hamiltonian.

Clearly, for any positive integer \( q \), \( |V(G)| = 3q + 2 \) and thus, \( \left\lfloor \frac{|V(G)|}{3} \right\rfloor = q + 1 \). Therefore, \( |V(G)| - |U| \) is odd and thus, \( |V'| = |V\backslash v_k| \) is even. By this then, the members of \( V' \) can be paired to form \( E(V') \).

**Proof.** It is easily verifiable that \( |E(V')| = |U| - 1 \). Now, let \( v_k \in V \) such that there is no such vertex \( v_j \in U \) such that \( v_kv_j \in E(G) \). Since \( V \subseteq N_G(u_i) \) for all \( u_i \in U \), then there exist \( u_a, u_b \in U \) such that \( u_kv_ku_b \) form a path \( P_3 \in G \). Clearly of all the \( q + 1 \) vertices in \( U \), two vertices \( u_a, v_b \) are already incident to \( v_k \). Now, we can imagine 'fusing' \( v_a, v_b \) into a single vertex \( v_{ab} \in U \) and therefore for the new \( U \), say, \( U' \backslash U' = q \). So for \( E(V') \), with \( |E(V')| = q \). The claim follows directly from Lemma 4 and Theorem 5.
3. Implication of the Results

It is clear from Theorem 5 that for every pair \( v_1, v_2 \in V \), \( d(v_1, v_2) \leq 2 \) and in many cases, equality holds. Since \( U \subset N_G(v_i), v_i \in V \), and \( v_i, v_{i+1} \in E(V) \), then \( d(v_i) \geq \frac{|V(G)|}{3} + 1 \). Thus \( d(v') + d(v'') \geq \frac{3}{2}(|V(G)| + 1) \) for cases where \( d(v', v'') = 2 \). Likewise, in Theorems 6 and 8, \( d(v') + d(v'') \geq \frac{3}{2}(|V(G)| + 2) \) and \( d(v') + d(v'') \geq \frac{3}{2}(|V(G)| + 4) \) respectively. This is a significant improvement over the result in Theorem 2 for this class of graphs. It is especially obvious as the order of \( G \) increases.

References


Deborah Olayide Ajayi
Department of Mathematics,
University of Ibadan,
Ibadan, Nigeria
email: olayide.ajayi@mail.ui.edu.ng; adelaidelajayi@yahoo.com

Tayo Charles Adefokun
Department of Computer and Mathematical Sciences,
Crawford University,
Nigeria
email: tayo.adefokun@gmail.com

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