Acta Universitatis Apulensis No. 41/2015
ISSN: 1582-5329 pp. 221-241
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.41.15

ABOUT THE EQUIVALENCE OF SOME CLASSICAL INEQUALITIES

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Abstract. The equivalence of the classical inequalities studied in [2], [4], [6], [7], [8]. follows from Jensen inequality as a property of the convex functions. Following a long way, but simple and general, in this paper we show that the equivalence of classical inequalities in finite dimensional case can be proved without using directly Jensen inequality, see also [5], [11].

2010 *Mathematics Subject Classification*: 26D15.

Keywords: Cauchy inequality, Root mean square inequality, Lyapunov inequality, Rearrangement inequality, Rado-Popoviciu inequality, Cauchy-Bunyakovski--Schwarz inequality, Young inequality, Bernoulli inequality, Maclaurin inequality, Rogers-Hölder inequality, Rogers inequality, Power mean inequality, Minkowski inequality.

1. Classical inequalities.

Euclid ordinary inequality. *If $a, b \in (0, +\infty)$, then*

$$ab \leq \left(\frac{a+b}{2}\right)^2. \tag{IOE}$$

with equality if and only if $a = b$.

Cauchy trivial inequality. *Let $n \in \mathbb{N}^*$ be any positive integer. If $a \in (0, +\infty)$, then*

$$a^{n/(n+1)} \leq \frac{1+na}{n+1} \tag{CTI}$$

with equality if and only if $a = 1$.

Cauchy elementary inequality. *Let $n \in \mathbb{N}^*$ be any positive integer and let $y_1, y_2, \ldots, y_n \in (0, +\infty)$ be any positive real numbers such that $y_1 y_2 \cdots y_n = 1$. Then*

$$y_1 + y_2 + \ldots + y_n \geq n \tag{CEI}$$

with equality if and only if $y_1 = y_2 = \ldots = y_n = 1$.

Cauchy mean inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer and let* $a_1, a_2, \ldots, a_n \in (0, +\infty)$ *be any positive real numbers. Then*

$$G_n = (a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n} = A_n; \tag{MGA}$$

and

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \leq (a_1 a_2 \ldots a_n)^{1/n} = G_n. \tag{MHG}$$

Equality in each inequality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Root mean square inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer and let* $a_1, a_2, \ldots, a_n \in (0, +\infty)$ *be any positive real numbers. Then*

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} = R_n \tag{RMS}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Rearrangement inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer and let* $z_1, z_2, \ldots, z_n \in (0, +\infty)$ *be any positive real numbers. If* $(w_1, w_2, \ldots, w_n)$ *is a permutation of* $(z_1, z_2, \ldots, z_n)$ *then*

$$z_1^2 + z_2^2 + \ldots + z_n^2 \geq z_1 w_1 + z_2 w_2 + \ldots + z_n w_n \tag{PIA}$$

and

$$\frac{z_1}{w_1} + \frac{z_2}{w_2} + \ldots + \frac{z_n}{w_n} \geq n. \tag{PIB}$$

Equality in each inequality holds if and only if $w_k = z_k$, *for all* $k = 1, 2, \ldots, n$.

Cauchy-Bunyakovski-Schwarz inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer. If* $a_k \in (0, +\infty)$ *and* $b_k \in (0, +\infty)$, *for all* $k = 1, 2, \ldots, n$, *then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right). \tag{CBS}$$

Equality holds if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}.$$

Bernoulli inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer and let* $\alpha \in (0, +\infty)$ *be any positive real number. Assume that* $x \in (-1, +\infty)$. *Then*

$$(1 + x)^n \geq 1 + nx. \tag{BIA}$$

If $\alpha \in [1, +\infty)$, *then*

$$(1+x)^\alpha \geq 1 + \alpha x. \tag{BIB}$$

If $\alpha \in (0, 1]$, *then*

$$(1+x)^\alpha \leq 1 + \alpha x. \tag{BIC}$$

Equality in each inequality holds if and only if $x = 0$.

Young inequality. *Assume that* $a, b \in (0, +\infty)$. *If* $\alpha \in (0, 1)$ *then*

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha) b. \tag{YIA}$$

Let $p, q \in (1, +\infty)$ *such that* $1/p + 1/q = 1$. *Then*

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q} \tag{YIB}$$

or equivalent

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{YIC}$$

Equality in each inequality holds if and only if $a = b$.

Rado-Popoviciu inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integers and let* $a_1, a_2, \ldots, a_n \in (0, +\infty)$ *be any positive real numbers. For all* $k = 1, 2, \ldots, n$, *we consider*

$$G_k = G(a_1, a_2, \ldots, a_k) = (a_1 a_2 \cdots a_n)^{1/k},$$

$$A_k = A(a_1, a_2, \ldots, a_k) = \frac{a_1 + a_2 + \ldots + a_k}{k}$$

and

$$H_k = H(a_1, a_2, \ldots, a_k) = \frac{k}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_k}}.$$

Then we have

$$0 = A_1 - G_1 \leq 2(A_2 - G_2) \leq 3(A_3 - G_3) \leq \ldots \leq n(A_n - G_n) \tag{RPA}$$

and

$$1 = \frac{G_1}{A_1} \geq \left(\frac{G_2}{A_2}\right)^2 \geq \ldots \geq \left(\frac{G_{n-1}}{A_{n-1}}\right)^{n-1} \geq \left(\frac{G_n}{A_n}\right)^n. \tag{RPB}$$

Equality in each inequality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Maclaurin inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer and let* $a_1, a_2, \ldots, a_n \in (0, +\infty)$ *be any positive real numbers. For all* $k = 1, 2, \ldots, n$, *we consider*

$$t_k = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} a_{j_1} a_{j_2} \cdots a_{j_k} \quad \text{and} \quad S_k = \frac{t_k}{C_n^k}.$$

Then

$$S_1 \geq S_2^{1/2} \geq S_3^{1/3} \geq \ldots \geq S_{n-1}^{1/(n-1)} \geq S_n^{1/n} \tag{MLI}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

Rogers-Hölder inequality. *Let* $n \in \mathbb{N}^*$ *be any positive interger. Assume that* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *are positive real numbers. Let* $p, q \in (1, +\infty)$ *such that* $1/p + 1/q = 1$. *Then*

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \tag{RHA}$$

with equality if and only if

$$\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \ldots = \frac{a_n^p}{b_n^q}.$$

If $t \in (1, +\infty)$, *then*

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k\right)^{1-1/t} \left(\sum_{k=1}^{n} a_k b_k^t\right)^{1/t} \tag{RHB}$$

with equality if and only if $b_1 = b_2 = \ldots = b_n$.

Rogers inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer. Assume that* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *are positive real numbers. Then*

$$b_1^{a_1} b_2^{a_2} \cdots b_n^{a_n} \leq \left(\frac{a_1 b_1 + a_2 b_2 + \ldots + a_n b_n}{a_1 + a_2 + \ldots + a_n}\right)^{a_1 + a_2 + \ldots + a_n}. \tag{RIA}$$

If $0 < r < s < t < +\infty$, *then*

$$\left(\sum_{k=1}^{n} a_k b_k^s\right)^{t-r} \leq \left(\sum_{k=1}^{n} a_k b_k^r\right)^{t-s} \left(\sum_{k=1}^{n} a_k b_k^t\right)^{s-r}. \tag{RIB}$$

Let $t_1, t_2, \ldots, t_n \in (0, +\infty)$ *be any positive real numbers such that* $t_1 + t_2 + \ldots + t_n = 1$. *Then*

$$b_1^{t_1} b_2^{t_2} \cdots b_n^{t_n} \leq t_1 b_1 + t_2 b_2 + \ldots + t_n b_n. \tag{RIC}$$

Equality in each inequality holds if and only if $b_1 = b_2 = \ldots = b_n$.

Lyapunov inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer. Assume that* $b_k \in (0, +\infty)$, *for all* $k = 1, 2, \ldots, n$. *If* $0 < r < s < t < +\infty$, *then*

$$\left(\sum_{k=1}^{n} b_k^s\right)^{t-r} \leq \left(\sum_{k=1}^{n} b_k^r\right)^{t-s} \left(\sum_{k=1}^{n} b_k^t\right)^{s-r} \tag{LI}$$

with equality if and only if $b_1 = b_2 = \ldots = b_n$.

Power mean inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer. Assume that* $x_k, \alpha_k \in (0, +\infty)$ *for all* $k = 1, 2, \ldots, n$. *The power mean* $M : \mathbb{R} \to \mathbb{R}$ *is defined by*

$$M(t) = \begin{cases} \left(\dfrac{\alpha_1 x_1^t + \alpha_2 x_2^t + \ldots + \alpha_n x_n^t}{\alpha_1 + \alpha_2 + \ldots + \alpha_n} \right)^{1/t} & , \quad t \in \mathbb{R}^* \\ (x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n})^{1/(\alpha_1 + \alpha_2 + \ldots + \alpha_n)} & , \quad t = 0. \end{cases}$$

If $s < t$, *then*

$$M(s) \le M(t), \tag{PMI}$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

Minkowski inequality. *Let* $n \in \mathbb{N}^*$ *be any positive integer. If* $p \in (1, +\infty)$ *and* $a_k, b_k \in (0, +\infty)$ *for all* $k = 1, 2, \ldots, n$, *then*

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p \right)^{1/p} \le \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p \right)^{1/p}. \tag{MI}$$

2. Proofs for equivalence of some classical inequalities.

Next we shall prove that the previous inequalities are true and equivalent without using directly Jensen inequality.

Lemma 1. *Let* $n \in \mathbb{N}^*$ *be any positive integer.*

(a). *For all real numbers* $x \in \mathbb{R}$ *we have*

$$x\big[nx^{n+1} - (n+1)x^n + 1\big] = (x-1)^2 \sum_{k=1}^{n} kx^k.$$

(b). *If* $x \in (0, +\infty)$, *then* $nx^{n+1} + 1 \ge (n+1)x^n$, *with equality if and only if* $x = 1$.

(c). *If* $a \in (0, +\infty)$, *then* $(n+1)a^{n/(n+1)} \le 1 + na$, *with equality if and only if* $a = 1$.

Proof. For $x \neq 1$ we have

$$\begin{aligned}\sum_{k=1}^{n} kx^k &= \sum_{k=1}^{n}\sum_{j=k}^{n} x^j = \sum_{k=1}^{n}(x^k + x^{k+1} + \ldots + x^n) \\ &= \sum_{k=1}^{n} x^k(1 + x + \ldots + x^{n-k}) = \sum_{k=1}^{n} x^k \frac{x^{n-k+1}-1}{x-1} \\ &= \frac{1}{x-1}\sum_{k=1}^{n}(x^{n+1} - x^k) = \frac{1}{x-1}\left(nx^{n+1} - \sum_{k=1}^{n} x^k\right) \\ &= \frac{1}{x-1}\left[nx^{n+1} - x\frac{x^n-1}{x-1}\right] = \frac{x}{(x-1)^2}\big[nx^{n+1} - (n+1)x^n + 1\big].\end{aligned}$$

Lemma 2. *The Euclid ordinary inequality* (IOE) *and the Cauchy mean inequality* (MGA) *are equivalent.*

Proof. (IOE) *implies* (MGA). [see [4], p.17] Assume that $b_k \in (0, +\infty)$ for all $k = 1, 2^1, \ldots, 2^n$. We have that $b_1 b_2 \leq \big((b_1 + b_2)/2\big)^2$, and so

$$b_1 b_2 b_3 b_4 \leq \left(\frac{b_1 + b_2}{2}\right)^2 \left(\frac{b_3 + b_4}{2}\right)^2 \leq \left(\frac{b_1 + b_2 + b_3 + b_4}{4}\right)^{2^2}.$$

Repeating the argument n times, we find

$$b_1 b_2 \cdots b_n b_{n+1} \cdots b_{2^n} \leq \left(\frac{b_1 + b_2 + \ldots + b_n + b_{n+1} + \ldots + b_{2^n}}{2^n}\right)^{2^n}.$$

Finally, taking $b_1 = a_1$, $b_2 = a_2, \ldots, b_n = a_n$ and

$$b_{n+1} = b_{n+2} = \ldots = b_{2^n} = \frac{a_1 + a_2 + \ldots + a_n}{n} = A_n$$

we obtain

$$\begin{aligned}a_1 a_2 \cdots a_n A_n^{2^n - n} &\leq \left(\frac{a_1 + a_2 + \ldots + a_n + (2^n - n)A_n}{2^n}\right)^{2^n} \\ &= \left(\frac{nA_n + (2^n - n)A_n}{2^n}\right)^{2^n} = A_n^{2^n}\end{aligned}$$

or equivalent $G_n^n = a_1 a_2 \cdots a_n \leq A_n^n$.

Remark 1. *We note the following elementary proof for the Cauchy-Bunyakovski-Schwarz inequality.*

Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \sum_{k=1}^{n} (a_k - b_k x)^2 = \sum_{k=1}^{n} a_k^2 - 2x \sum_{k=1}^{n} a_k b_k + x^2 \sum_{k=1}^{n} b_k^2.$$

We observe that $f(x) \geq 0$ *for all* $x \in \mathbb{R}$. *For*

$$x_0 = \left(\sum_{j=1}^{n} a_j b_j \right) \Big/ \left(\sum_{j=1}^{n} b_j \right)$$

we have that the inequality $f(x_0) \geq 0$ *is equivalent to (CBS) inequality.*

Remark 2. *The Cauchy mean inequality* (MGA) *and the Cauchy mean inequality* (MHG) *are equivalent.*

We observe that $H(a_1, a_2, \dots, a_n) \leq G(a_1, a_2, \dots, a_n)$ *is equivalent to*

$$G\left(1/a_1, 1/a_2, \dots, 1/a_n\right) \leq A\left(1/a_1, 1/a_2, \dots, 1/a_n\right).$$

Similarly, we have that $H\left(1/a_1, 1/a_2, \dots, 1/a_n\right) \leq G\left(1/a_1, 1/a_2, \dots, 1/a_n\right)$ *is equivalent to* $G(a_1, a_2, \dots, a_n) \leq A(a_1, a_2, \dots, a_n)$.

Lemma 3. *The Cauchy mean inequality* (MGA) *and the Cauchy elementary inequality* (CEI) *are equivalent.*

Proof. (MGA) *implies* (CEI). Since $y_1 y_2 \dots y_n = 1$, from the Cauchy mean inequality (MGA), we obtain $y_1 + y_2 + \dots + y_n \geq n(y_1 y_2 \cdots y_n)^{1/n} = n$.

(CEI) *implies* (MGA). We put $y_k = a_k/G_n$, for all $k = 1, 2, \dots, n$. Then $y_1 y_2 \dots y_n = 1$. From the Cauchy elementary inequality (CEI), we have that $a_1/G_n + a_2/G_n + \dots + a_n/G_n \geq n$, or equivalent $A_n \geq G_n$.

Lemma 4. *The Cauchy mean inequality* (MGA)*, the Cauchy trivial inequality* (CTI) *and the Bernoulli inequality* (BIA) *are equivalent.*

Proof. (MGA) *implies* (CTI). For $x_1 = x_2 = \dots = x_n = a$ and $x_{n+1} = 1$ we obtain that $G(x_1, x_2, \dots, x_n, x_{n+1}) \leq A(x_1, x_2, \dots, x_n, x_{n+1})$ is equivalent to $a^{n/(n+1)} \leq (1 + na)/(n + 1)$.

(MGA) *implies* (BIA). Assume that $n \geq 2$. Since $x \in (-1, +\infty)$ we have that $y = (1 + x)^n \in (0, +\infty)$. We put $a_1 = y$, $a_2 = a_3 = \dots = a_n = 1$. Using the Cauchy mean inequality (MGA) we obtain $y + n - 1 \geq n \sqrt[n]{y}$, or equivalent $(1 + x)^n = y \geq 1 + n(\sqrt[n]{y} - 1) = 1 + nx$.

(BIA) *implies* (MGA). (see [1],p.398) Assume that $n \geq 2$. Let $k \in \{2, 3, \ldots, n\}$. From the Bernoulli inequality (BIA), for $x = A_k/A_{k-1}$, we have

$$\left(\frac{A_k}{A_{k-1}}\right)^k \geq 1 + k\left(\frac{A_k}{A_{k-1}} - 1\right) = \frac{kA_k - (k-1)A_{k-1}}{A_{k-1}} = \frac{a_k}{A_{k-1}},$$

or equivalent $A_k^k \geq a_k A_{k-1}^{k-1}$. Therefore

$$A_n^n \geq a_n A_{n-1}^{n-1} \geq a_n a_{n-1} A_{n-2}^{n-2} \geq \ldots \geq a_n a_{n-1} \ldots a_2 A_1^1 = a_n a_{n-1} \ldots a_2 a_1.$$

(CTI) *implies* (BIA). Since $x \in (-1, +\infty)$, we have that $a = 1/(1+x)^{n+1} \in (0, +\infty)$. Applying the Cauchy trivial inequality (CTI) we obtain that $(1+x)^{n+1} \geq 1 + (n+1)x$.

Lemma 5. *The Cauchy-Bunyakovski-Schwarz inequality* (CBS) *and the rearrangement inequality* (PIA) *are equivalent.*

Proof. (CBS) *implies* (PIA). Since $z_1^2 + z_2^2 + \ldots + z_n^2 = w_1^2 + w_2^2 + \ldots + w_n^2$, using Cauchy-Bunyakovski-Schwarz inequality (CBS), we obtain

$$\sum_{k=1}^{n} z_k w_k \leq \left(\sum_{k=1}^{n} z_k^2\right)^{1/2} \left(\sum_{k=1}^{n} w_k^2\right)^{1/2} = \sum_{k=1}^{n} z_n^2.$$

(PIA) *implies* (CBS). (see [12]) Define $S = (a_1^2 + a_2^2 + \ldots + a_n^2)^{1/2}$ and $T = (b_1^2 + b_2^2 + \ldots + b_n^2)^{1/2}$. We put $y_k = a_k/S$ and $y_{n+k} = b_k/T$, for all $k = 1, 2, \ldots, n$. Observe that $(y_{n+1}, y_{n+2}, \ldots, y_{2n}, y_1, y_2, \ldots, y_n)$ is a permutation of $(y_1, y_2, \ldots, y_n, y_{n+1}, y_{n+2}, \ldots, y_{2n})$. Applying the rearrangement inequality (PIA), we have

$$\begin{aligned}
2 &= \frac{a_1^2 + a_2^2 + \ldots + a_n^2}{S^2} + \frac{b_1^2 + b_2^2 + \ldots + b_n^2}{T^2} \\
&= y_1^2 + y_2^2 + \ldots + y_n^2 + y_{n+1}^2 + y_{n+2}^2 + \ldots + y_{2n}^2 \\
&\geq y_1 y_{n+1} + y_2 y_{n+2} + \ldots + y_n y_{2n} + y_{n+1} y_1 + y_{n+2} y_2 + \ldots + y_{2n} y_n \\
&= \frac{2(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)}{ST}.
\end{aligned}$$

Lemma 6. *The Cauchy mean inequality* (MGA) *and the rearrangement inequality* (PIB) *are equivalent.*

Proof. (MGA) *implies* (PIB). Since $z_1 z_2 \cdots z_n = w_1 w_2 \cdots w_n$, using the Cauchy mean inequality (MGA), we obtain

$$\frac{z_1}{w_1} + \frac{z_2}{w_2} + \ldots + \frac{z_n}{w_n} \geq n \sqrt[n]{\frac{z_1}{w_1} \cdot \frac{z_2}{w_2} \cdots \frac{z_n}{w_n}} = n.$$

(PIB) *implies* (MGA). (see [12]) Now we put $G_n = (a_1 a_2 \cdots a_n)^{1/n}$ and

$$y_1 = \frac{a_1}{G_n}, \quad y_2 = \frac{a_1 a_2}{G_n^2}, \ldots, y_{n-1} = \frac{a_1 a_2 \cdots a_{n-1}}{G_n^{n-1}}, \quad y_n = \frac{a_1 a_2 \cdots a_n}{G_n^n} = 1.$$

Then $(y_n, y_1, y_2, \ldots, y_{n-2}, y_{n-1})$ is a permutation of $(y_1, y_2, \ldots, y_{n-1}, y_n)$. Applying the rearrangement inequality (PIB) we have

$$n \leq \frac{y_1}{y_n} + \frac{y_2}{y_1} + \ldots + \frac{y_{n-1}}{y_{n-2}} + \frac{y_n}{y_{n-1}} = \frac{a_1}{G_n} + \frac{a_2}{G_n} + \ldots + \frac{a_n}{G_n}$$

which is equivalent to $G_n = (a_1 a_2 \cdots a_n)^{1/n} \leq \dfrac{a_1 + a_2 + \ldots + a_n}{n} = A_n$.

Remark 3. *Let* $\alpha \in (0,1)$ *such that* $s = \alpha t + (1-\alpha) r$. *Then* $\alpha = (s-r)/(t-r)$ *and* $1 - \alpha = (t-s)/(t-r)$. *The Rogers inequality* (RIB) *can be written in the following equivalent form*

$$\sum_{k=1}^{n} a_k \left(b_k^t\right)^{\alpha} \left(b_k^r\right)^{1-\alpha} \leq \left(\sum_{k=1}^{n} a_k b_k^r\right)^{1-\alpha} \left(\sum_{k=1}^{n} a_k b_k^t\right)^{\alpha}. \tag{RBI}$$

Similarly, the Lyapunov inequality (LI) *can be written in the following equivalent form*

$$\sum_{k=1}^{n} \left(b_k^t\right)^{\alpha} \left(b_k^r\right)^{1-\alpha} \leq \left(\sum_{k=1}^{n} b_k^r\right)^{1-\alpha} \left(\sum_{k=1}^{n} b_k^t\right)^{\alpha}. \tag{IL}$$

Lemma 7. *The Rogers inequality* (RIB)*, the Lyapunov inequality* (LI) *and the Cauchy-Bunyakovski-Schwarz inequality* (CBS) *are equivalent.*

Proof. (RIB) *implies* (LI). For $a_1 = a_2 = \ldots = a_n = 1$ from (RIB) we obtain (LI).

(LI) *implies* (RIB). From the Lyapunov inequality (IL), we get

$$\begin{aligned} \sum_{k=1}^{n} a_k b_k^s &= \sum_{k=1}^{n} \left(\left(a_k^{1/t} b_k\right)^t\right)^{\alpha} \left(\left(a_k^{1/r} b_k\right)^r\right)^{1-\alpha} \\ &\leq \left(\sum_{k=1}^{n} \left(a_k^{1/t} b_k\right)^t\right)^{\alpha} \left(\sum_{k=1}^{n} \left(a_k^{1/r} b_k\right)^r\right)^{1-\alpha} \\ &= \left(\sum_{k=1}^{n} a_k b_k^t\right)^{\alpha} \left(\sum_{k=1}^{n} a_k b_k^r\right)^{1-\alpha}. \end{aligned}$$

(RIB) *implies* (CBS). For $\alpha = 1/2$ the Rogers inequality (RIB) can be written in the following form

$$\sum_{k=1}^{n} a_k b_k^{t/2} b_k^{r/2} \leq \left(\sum_{k=1}^{n} a_k b_k^t \right)^{1/2} \left(\sum_{k=1}^{n} a_k b_k^r \right)^{1/2} \tag{1}$$

Assume that $x_k \in (0, +\infty)$ and $y_k \in (0, +\infty)$, for all $k = 1, 2, \ldots, n$. We consider the function $f : [0, +\infty) \to \mathbb{R}$,

$$f(u) = \sum_{k=1}^{n} x_k^{2u} y_k^{2(1-u)} = \sum_{k=1}^{n} y_k^2 \left(x_k^2 y_k^{-2} \right)^u.$$

Let $u, v \in [0, +\infty)$. The inequality (1) implies that

$$\begin{aligned} f\left(\frac{u+v}{2} \right) &= \sum_{k=1}^{n} y_k^2 \left(x_k^2 y_k^{-2} \right)^{u/2} \left(x_k^2 y_k^{-2} \right)^{v/2} \\ &\leq \left(\sum_{k=1}^{n} y_k^2 \left(x_k^2 y_k^{-2} \right)^u \right)^{1/2} \left(\sum_{k=1}^{n} y_k^2 \left(x_k^2 y_k^{-2} \right)^v \right)^{1/2} \\ &= f(u)^{1/2} f(v)^{1/2}. \end{aligned}$$

Let $t \in [0, 1]$. It follows that

$$f\left(\frac{1}{2} \right) = f\left(\frac{t}{2} + \frac{1-t}{2} \right) \leq f(t)^{1/2} f(1-t)^{1/2}.$$

Since $f : [0, +\infty) \to \mathbb{R}$ is continuous we have

$$f\left(\frac{1}{2} \right) \leq \lim_{t \nearrow 1} f(t)^{1/2} f(1-t)^{1/2} = f(1)^{1/2} f(0)^{1/2}.$$

This inequality is the Cauchy-Bunyakovski-Schwarz inequality (CBS)

$$\sum_{k=1}^{n} x_k y_k = f\left(\frac{1}{2} \right) \leq f(1)^{1/2} f(0)^{1/2} = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} \left(\sum_{k=1}^{n} y_k^2 \right)^{1/2}.$$

(CBS) *implies* (LI). Now we consider the function $g : [0, +\infty) \to \mathbb{R}$,

$$g(u) = \sum_{k=1}^{n} b_k^u.$$

Let $u, v \in [0, +\infty)$. The Cauchy-Bunyakovsky-Schwarz inequality (CBS) implies that

$$g\left(\frac{u+v}{2}\right) = \sum_{k=1}^{n} b_k^{u/2} b_k^{v/2} \le \left(\sum_{k=1}^{n} b_k^{u}\right)^{1/2} \left(\sum_{k=1}^{n} b_k^{v}\right)^{1/2} = g(u)^{1/2} g(v)^{1/2}.$$

Suppose that $u_1, u_2, u_3, u_4 \in [0, +\infty)$. Let $v_1 = (u_1 + u_2)/2$ and $v_2 = (u_3 + u_4)/2$. Then

$$\begin{aligned} g\left(\frac{u_1+u_2+u_3+u_4}{2^2}\right) &= g\left(\frac{v_1+v_2}{2}\right) \le g(v_1)^{1/2} g(v_2)^{1/2} \\ &\le g(u_1)^{1/4} g(u_2)^{1/4} g(u_3)^{1/4} g(u_4)^{1/4}. \end{aligned}$$

If $u_1, u_2, \ldots, u_{2^n} \in [0, +\infty)$, repeating the argument n times, we find

$$g\left(\frac{u_1+u_2+\ldots+u_{2^n}}{2^n}\right) \le g(u_1)^{1/2^n} g(u_2)^{1/2^n} \cdots g(u_{2^n})^{1/2^n}. \tag{2}$$

Let $v_1, v_2, \ldots, v_n \in [0, +\infty)$. Taking $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$, $u_{n+1} = u_{n+2} = \ldots = u_{2^n} = (v_1 + v_2 + \ldots + v_n)/n$ and applying (2), we find

$$\begin{aligned} g\left(\frac{v_1+v_2+\ldots+v_n}{n}\right) &= g\left(\frac{v_1+v_2+\ldots+v_n+(2^n-n)\dfrac{v_1+v_2+\ldots+v_n}{n}}{2^n}\right) \\ &= g\left(\frac{u_1+u_2+\ldots+u_n+u_{n+1}+\ldots+u_{2^n}}{2^n}\right) \\ &\le \big(g(u_1)g(u_2)\cdots g(u_n)g(u_{n+1})\cdots g(u_{2^n})\big)^{1/2^n} \\ &= \big(g(v_1)g(v_2)\cdots g(v_n)\big)^{1/2^n} g\left(\frac{v_1+v_2+\ldots+v_n}{n}\right)^{(2^n-n)/2^n} \end{aligned}$$

or equivalent

$$g\left(\frac{v_1+v_2+\ldots+v_n}{n}\right) \le g(v_1)^{1/n} g(v_2)^{1/n} \cdots g(v_n)^{1/n}. \tag{3}$$

Assume that $n \ge 2$. Let $k \in \{1, 2, \ldots, n-1\}$. For $v_1 = v_2 = \ldots = v_k = x$ and $v_{k+1} = v_{k+2} = \ldots = v_n = y$ the inequality (3) becomes

$$g\left(\frac{k}{n}x + \left(1 - \frac{k}{n}\right)y\right) \le g(x)^{k/n} g(y)^{1-k/n}. \tag{4}$$

Since any rational number $d \in (0,1)$ has a form k/n, our inequality (4) holds, for all $x, y \in [0,+\infty)$ and for any rational $d \in (0,1)$. If $\beta \in (0,1)$, then there exists a rational sequence $(d_j)_j \subset (0,1) \cap \mathbb{Q}$ convergent to β. Then, since $g : [0,+\infty) \to \mathbb{R}$ is continuous we have

$$g(\beta x + (1-\beta)y) = \lim_{j\to+\infty} g(d_j x + (1-d_j)y) \le \lim_{j\to+\infty} g(x)^{d_j} g(y)^{1-d_j} = g(x)^{\beta} g(y)^{1-\beta}.$$

Now we obtain the Lyapunov inequality (IL),

$$\begin{aligned}\sum_{k=1}^{n} \left(b_k^t\right)^{\alpha} \left(b_k^r\right)^{1-\alpha} &= \sum_{k=1}^{n} b_k^{\alpha t + (1-\alpha) r} = g(\alpha t + (1-\alpha) r) \\ &\le g(t)^{\alpha} g(r)^{1-\alpha} = \left(\sum_{k=1}^{n} b_k^t\right)^{\alpha} \left(\sum_{k=1}^{n} b_k^r\right)^{1-\alpha}.\end{aligned}$$

Lemma 8. *The Cauchy mean inequality* (MGA) *and the Rado-Popoviciu inequality* (RPB) *are equivalent.*

Proof. (RPB) *implies* (MGA). It is clear that from $1 \ge (G_n/A_n)^n$ we obtain $A_n \ge G_n$.

(MGA) *implies* (RPB). Assume that $n \ge 2$. Let $k \in \{1, 2, \ldots, n-1\}$. Since $G_{k+1}^{k+1} = a_{k+1} G_k^k$ and $(k+1)A_{k+1} - kA_k = a_{k+1}$, we observe that the inequality

$$\left(\frac{G_k}{A_k}\right)^k \ge \left(\frac{G_{k+1}}{A_{k+1}}\right)^{k+1}$$

can be written in the following equivalent form

$$\left(\frac{kA_k + a_{k+1}}{k+1}\right)^{k+1} \ge a_{k+1} A_k^k.$$

But this follows from the Cauchy mean inequality (MGA),

$$A(A_k, A_k, \ldots, A_k, a_{k+1}) \ge G(A_k, A_k, \ldots, A_k, a_{k+1}).$$

Lemma 9. *The Cauchy mean inequality* (MGA)*, the Cauchy trivial inequality* (CTI) *and the Rado-Popoviciu inequality* (RPA) *are equivalent.*

Proof. (RPA) *implies* (MGA). It is clear that from $0 \le n(A_n - G_n)$ we obtain $A_n \ge G_n$.

(CTI) *implies* (RPA). (see [9],p.18) Assume that $n \ge 2$. Let $k \in \{1, 2, \dots, n-1\}$. From the trivial inequality (CTI), for $a = G_k/a_{k+1}$, we have

$$\left(\frac{G_k}{a_{k+1}}\right)^k \le \left(\frac{a_{k+1} + kG_k}{(1+k)a_{k+1}}\right)^{k+1}.$$

It follows that

$$\left(\frac{G_{k+1}}{a_{k+1}}\right)^{k+1} = \frac{G_k^k a_{k+1}}{a_{k+1}^{k+1}} \le \left(\frac{a_{k+1} + kG_k}{(1+k)a_{k+1}}\right)^{k+1}$$

or equivalent $(k+1)G_{k+1} \le a_{k+1} + kG_k$. Since $a_{k+1} = (k+1)A_{k+1} - kA_k$ we obtain

$$(k+1)G_{k+1} \le (k+1)A_{k+1} - kA_k + kG_k \Leftrightarrow k(A_k - G_k) \le (k+1)(A_{k+1} - G_{k+1}).$$

Lemma 10. *The Rogers-Hölder inequality* (RHA) *and the Minkowski inequality* (MI) *are equivalent.*

Proof. (RHA) *implies* (MI). Let $p, q \in (1, +\infty)$ such that $1/p + 1/q = 1$. We have the followinf identity

$$S := \sum_{k=1}^{n} (a_k + b_k)^p = \sum_{k=1}^{n} a_k (a_k + b_k)^{p-1} + \sum_{k=1}^{n} b_k (a_k + b_k)^{p-1}.$$

Since $q(p-1) = p$, applying the Rogers-Hölder inequality (RHA), we obtain

$$S \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} (a_k + b_k)^{q(p-1)}\right)^{1/q} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p} \left(\sum_{k=1}^{n} (a_k + b_k)^{q(p-1)}\right)^{1/q}$$
$$= \left(\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\right) \left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/q}$$

or equivalent

$$S \le \left(\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\right) S^{1/q}.$$

(MI) *implies* (RHA). (see [7]) Let $p, q \in (1, +\infty)$ such that $1/p + 1/q = 1$ and $a, b, t \in (0, +\infty)$. From the Bernoulli inequality (BIB), we have $(1 + tb/a)^q \ge 1 + qtb/a$

or equivalent $(a+tb)^q \geq a^q + qtba^{q-1}$. Let $k \in \{1, 2, \ldots, n\}$. For $a = a_k^{1/(q-1)}$ and $b = b_k$ we obtain

$$qtb_k a_k \leq \left[a_k^{1/(q-1)} + tb_k\right]^q - a_k^{q/(q-1)}.$$

Then, summing over k from 1 to n, we obtain

$$qt\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{n}\left[a_k^{1/(q-1)} + tb_k\right]^q - \sum_{k=1}^{n} a_k^{q/(q-1)}.$$

Since $p = q(q-1)$, from the Minkowski inequality (MI), we have

$$qt\sum_{k=1}^{n} a_k b_k \leq \left[\left(\sum_{k=1}^{n} a_k^{q/(q-1)}\right)^{1/q} + t\left(\sum_{k=1}^{n} b_k^q\right)^{1/q}\right]^q - \sum_{k=1}^{n} a_k^{q/(q-1)}$$

or equivalent

$$qt\sum_{k=1}^{n} a_k b_k \leq \left[\left(\sum_{k=1}^{n} a_k^p\right)^{1/q} + t\left(\sum_{k=1}^{n} b_k^q\right)^{1/q}\right]^q - \sum_{k=1}^{n} a_k^p.$$

We consider

$$A = \sum_{k=1}^{n} a_k^p \text{ and } B = \left(\sum_{k=1}^{n} b_k^p\right)^{1/q}.$$

Now consider the function $f : [0, +\infty) \to \mathbb{R}, f(s) = (A^{1/q} + sB)^q$. It follows that

$$q\sum_{k=1}^{n} a_k b_k \leq \frac{f(t) - f(0)}{t},\ t \in (0, +\infty).$$

Then

$$q\sum_{k=1}^{n} a_k b_k \leq \lim_{t\searrow 0}\frac{f(t) - f(0)}{t} = f'(0) = qBA^{(q-1)/q} = qBA^{1/p}.$$

Lemma 11. *The Cauchy mean inequality* (MGA) *and the Maclaurin inequality* (MLI) *are equivalent.*

Proof. (MLI) *implies* (MGA). Indeed, we observe that

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} = \frac{t_1}{C_n^1} = S_1 \geq S_n^{1/n} = \left(\frac{t_n}{C_n^n}\right)^{1/n}$$
$$= (a_1 a_2 \cdots a_n)^{1/n} = G_n.$$

(MGA) *implies* (MLI). (see [10], p.36) Consider the function $h : \mathbb{R} \to \mathbb{R}$

$$h(x) = x^n - t_1 x^{n-1} + t_2 x^{n-2} - \ldots + (-1)^{n-1} t_{n-1} x + (-1)^n t_n.$$

Let $k \in \{1, 2, \ldots, n\}$ and let $x \in \mathbb{R}$. A computation shows that

$$h^{(n-k)}(x) = A_n^{n-k} x^k - t_1 A_{n-1}^{n-k} x^{k-1} + \ldots + (-1)^{k-1} t_{k-1} A_{n-k+1}^{n-k} x + (-1)^k t_k A_{n-k}^{n-k}.$$

From the Viète relations, the roots of equation $h(x) = 0$ are $a_1, a_2, \ldots, a_n$. It follows that the roots of equation $h^{(n-k)}(x) = 0$ are all positive real numbers. Let $x_1, x_2, \ldots, x_k \in (0, +\infty)$ the roots of equation $h^{(n-k)}(x) = 0$. We put $c_j = x_1 x_2 \cdots x_k / x_j$, for all $j = 1, 2, \ldots, k$. Now, using the Cauchy mean inequality (MGA), we obtain

$$(c_1 c_2 \cdots c_k)^{1/k} \leq \frac{c_1 + c_2 + \ldots + c_k}{k}$$

or equivalent

$$(x_1 x_2 \cdots x_k)^{(k-1)/k} \leq \frac{1}{k} \sum_{1 \leq j_1 < j_2 < \ldots < j_{k-1} \leq k} x_{j_1} x_{j_2} \cdots x_{j_{k-1}}.$$

Since, from the Viète relations, $x_1 x_2 \cdots x_k = t_k A_{n-k}^{n-k} / A_n^{n-k} = t_k / C_n^k$ and

$$\sum_{1 \leq j_1 < j_2 < \ldots < j_{k-1} \leq k} x_{j_1} x_{j_2} \cdots x_{j_{k-1}} = t_{k-1} A_{n-k+1}^{n-k} / A_n^{n-k} = k t_{k-1} / C_n^{k-1}$$

we find that $(t_k / C_n^k)^{(k-1)/k} \leq t_{k-1} / C_n^{k-1}$.

Lemma 12. *The Rogers inequality* (RIB)*, the power mean inequality* (PMI)*, the Rogers-Hölder inequality* (RHB) *and the Rogers-Hölder inequality* (RHA) *are equivalent.*

Proof. (RIB) *implies* (PMI). Assume that $0 < s < t$. Let $r \in (0, +\infty)$ such that $0 < r < s < t$. Using the Rogers inequality (RIB) we have

$$\left(\sum_{k=1}^n \alpha_k x_k^s \right)^{t-r} \leq \left(\sum_{k=1}^n \alpha_k x_k^t \right)^{s-r} \left(\sum_{k=1}^n \alpha_k x_k^r \right)^{t-s}.$$

For $r \searrow 0$ we have that

$$\left(\sum_{k=1}^n \alpha_k x_k^s \right)^t \leq \left(\sum_{k=1}^n \alpha_k x_k^t \right)^s \left(\sum_{k=1}^n \alpha_k \right)^{t-s},$$

or equivalent

$$\left(M(s)^s \sum_{k=1}^{n} \alpha_k\right)^t \leq \left(M(t)^t \sum_{k=1}^{n} \alpha_k\right)^s \left(\sum_{k=1}^{n} \alpha_k\right)^{t-s}.$$

Using this inequality we obtain $M(s)^{st} \leq M(t)^{ts}$. This inequality becomes $M(s) \leq M(t)$. Now assume that $s < t < 0$ and let $r \in (0, +\infty)$ such that $0 < r < -t < -s$. Using the Rogers inequality (RIB), we have

$$\left(\sum_{k=1}^{n} \alpha_k \left(\frac{1}{x_k}\right)^{-t}\right)^{-s-r} \leq \left(\sum_{k=1}^{n} \alpha_k \left(\frac{1}{x_k}\right)^{r}\right)^{-s+t} \left(\sum_{k=1}^{n} \alpha_k \left(\frac{1}{x_k}\right)^{-s}\right)^{-t-r}.$$

For $r \searrow 0$, we have that

$$\left(\sum_{k=1}^{n} \alpha_k x_k^t\right)^{-s} \leq \left(\sum_{k=1}^{n} \alpha_k\right)^{t-s} \left(\sum_{k=1}^{n} \alpha_k x_k^s\right)^{-t}$$

or equivalent

$$\left(M(t)^t \sum_{k=1}^{n} \alpha_k\right)^{-s} \leq \left(\sum_{k=1}^{n} \alpha_k\right)^{t-s} \left(M(s)^s \sum_{k=1}^{n} \alpha_k\right)^{-t}.$$

Using this inequality, we obtain $M(t)^{-st} \leq M(s)^{-st}$. This inequality becomes $M(s) \leq M(t)$. Finally, we assume that $s < 0 < t$. Let $u, v \in \mathbb{R}^*$ such that $s < u < 0 < v < t$. Then $M(s) \leq M(u)$ and $M(v) \leq M(t)$. It follows that $M(s) \leq \lim_{u \nearrow 0} M(u) = M(0) = \lim_{v \searrow 0} M(v) \leq M(t)$.

(PMI) *implies* (RHB). Assume that $1 < t < +\infty$. Then the inequality $M(1) \leq M(t)$ can be written as

$$\frac{\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n}{\alpha_1 + \alpha_2 + \ldots + \alpha_n} \leq \left(\frac{\alpha_1 x_1^t + \alpha_2 x_2^t + \ldots + \alpha_n x_n^t}{\alpha_1 + \alpha_2 + \ldots + \alpha_n}\right)^{1/t}$$

or equivalent

$$\sum_{k=1}^{n} \alpha_k x_k \leq \left(\sum_{k=1}^{n} \alpha_k\right)^{1-1/t} \left(\sum_{k=1}^{n} \alpha_k x_k^t\right)^{1/t}.$$

(RHB) *implies* (RHA). Let $t = q \in (1, +\infty)$. Then $1/p = 1 - 1/t$. Since

$p+q-pq=0$, applying the Rogers-Hölder inequality (RHB), we have

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k^p (a_k^{1-p} b_k) \le \left(\sum_{k=1}^{n} a_k^p\right)^{1-1/q} \left(\sum_{k=1}^{n} a_k^p (a_k^{1-p} b_k)^q\right)^{1/q}$$
$$= \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}.$$

(RHA) *implies* (RIB). We prove that (RHA) implies (RBI). For $\alpha = 1/p$ and $1-\alpha = 1/q$, applying the Rogers-Hölder inequality (RHA), we have

$$\sum_{k=1}^{n} a_k (b_k^t)^{\alpha} (b_k^r)^{1-\alpha} = \sum_{k=1}^{n} (a_k^{1/t} b_k)^{t\alpha} (a_k^{1/r} b_k)^{r(1-\alpha)}$$
$$\le \left(\sum_{k=1}^{n} (a_k^{1/t} b_k)^{p\alpha t}\right)^{1/p} \left(\sum_{k=1}^{n} (a_k^{1/r} b_k)^{qr(1-\alpha)}\right)^{1/q}$$
$$= \left(\sum_{k=1}^{n} a_k b_k^t\right)^{\alpha} \left(\sum_{k=1}^{n} a_k b_k^r\right)^{1-\alpha}.$$

Remark 4. *The Young inequality* (YIA)*, the Young inequality* (YIB) *and the Young inequality* (YIC) *are equivalent. The Bernoulli inequality* (BIB) *and the Bernoulli inequality* (BIC) *are equivalent. The Rogers inequality* (RIA) *and the Rogers inequality* (RIC) *are equivalent. Finally, we note that the Bernoulli inequality* (BIB) *implies the Bernoulli inequality* (BIA).

Lemma 13. *The Rogers inequality (*RIC) *and the Young inequality* (YIA) *are equivalent.*

Proof. (RIC) *implies* (YIA). Trivial.

(YIA) *implies* (RIC). We proceed by induction. Let $n \in \mathbb{N}^*$ with $n \ge 2$ and $b_1, b_2, \ldots, b_n, b_{n+1} \in (0,+\infty)$. Suppose that $t_1, t_2, \ldots, t_n, t_{n+1} \in (0,+\infty)$ are such that $t_1+t_2+\ldots+t_n+t_{n+1}=1$. Let $s = 1-t_{n+1} = t_1+t_2+\ldots+t_n \in (0,1)$. Thus, by the inductive assumption, we have

$$b_1^{t_1} b_2^{t_2} \cdots b_n^{t_n} b_{n+1}^{t_{n+1}} = \left(b_1^{t_1/s} b_2^{t_2/s} \cdots b_n^{t_n/s}\right)^s b_{n+1}^{t_{n+1}} \le s b_1^{t_1/s} b_2^{t_2/s} \cdots b_n^{t_n/s} + t_{n+1} b_{n+1}$$

$$\le s\left[\frac{t_1}{s} b_1 + \frac{t_2}{s} b_2 + \ldots + \frac{t_n}{s} b_n\right] + t_{n+1} b_{n+1} = t_1 b_1 + t_2 b_2 + \ldots + t_n b_n + t_{n+1} b_{n+1}.$$

Lemma 14. *The Bernoulli inequality* (BIC) *and the Young inequality* (YIA) *are equivalent.*

Proof. We observe that the Young inequality (YIA) can be written in the following equivalent form $(a/b)^\alpha \le 1 - \alpha(1 - a/b)$.

(BIC) *implies* (YIA). For $1 + x = a/b \in (0, +\infty)$, from the Bernoulli inequality (BIC) we have $(a/b)^\alpha \le 1 - \alpha(1 - a/b)$.

(YIA) *implies* (BIC). For $b = 1$ and $a = x + 1 \in (0, +\infty)$, from the Young inequality (YIA) we have $(1 + x)^\alpha \le 1 + \alpha x$.

Lemma 15. *The Bernoulli inequality* (BIB)*, the power mean inequality* (PMI)*, the Rogers inequality* (RIA)*, the Cauchy mean inequality* (MGA) *and the Young inequality* (YIA) *are equivalent.*

Proof. (BIB) *implies* (PMI). (see [3], p.109) Let $k \in \{1, 2, \ldots, n\}$. Assume that $0 < s < t$. Since $t/s > 1$, using the Bernoulli inequality (BIB) we have

$$\alpha_k \left(\frac{x_k}{M(s)}\right)^t = \alpha_k \left[\left(\frac{x_k}{M(s)}\right)^s\right]^{t/s} \ge \alpha_k + \frac{t\alpha_k}{s}\left[\frac{x_k^s}{M(s)^s} - 1\right].$$

Now, summing over k from 1 to n we obtain

$$\frac{1}{M(s)^t}\sum_{k=1}^n \alpha_k x_k^t \ge \sum_{k=1}^n \alpha_k + \frac{t}{sM(s)^s}\sum_{k=1}^n \alpha_k x_k^s - \frac{t}{s}\sum_{k=1}^n \alpha_k$$

or equivalent

$$\left(\frac{M(t)}{M(s)}\right)^t \sum_{k=1}^n \alpha_k \ge \sum_{k=1}^n \alpha_k + \frac{t}{sM(s)^s} M(s)^s \sum_{k=1}^n \alpha_k - \frac{t}{s}\sum_{k=1}^n \alpha_k.$$

This implies that $M(t)^t \ge M(s)^t$, that is $M(t) \ge M(s)$. Now assume that $s < t < 0$. Let $k \in \{1, 2, \ldots, n\}$. Since $s/t > 1$, using the Bernoulli inequality (BIB) we have

$$\alpha_k \left(\frac{x_k}{M(t)}\right)^s = \alpha_k \left[\left(\frac{x_k}{M(t)}\right)^t\right]^{s/t} \ge \alpha_k + \frac{s\alpha_k}{t}\left[\frac{x_k^t}{M(t)^t} - 1\right],$$

and summing over k from 1 to n we obtain

$$\frac{1}{M(t)^s}\sum_{k=1}^n \alpha_k x_k^s \ge \sum_{k=1}^n \alpha_k + \frac{s}{tM(t)^t}\sum_{k=1}^n \alpha_k x_k^t - \frac{s}{t}\sum_{k=1}^n \alpha_k$$

or equivalent

$$\left(\frac{M(s)}{M(t)}\right)^s \sum_{k=1}^n \alpha_k \geq \sum_{k=1}^n \alpha_k + \frac{s}{tM(t)^t} M(t)^t \sum_{k=1}^n \alpha_k - \frac{s}{t} \sum_{k=1}^n \alpha_k.$$

This implies that $M(t)^{-s} \geq M(s)^{-s}$, that is $M(t) \geq M(s)$. Finally, we assume that $s < 0 < t$. Let $u, v \in \mathbb{R}^*$ such that $s < u < 0 < v < t$. Then $M(s) \leq \lim_{u \nearrow 0} M(u) = M(0) = \lim_{v \searrow 0} M(v) \leq M(t)$.

(PMI) *implies* (RIA). We observe that the inequality $M(0) \leq M(1)$ is equivalent to the Rogers inequality (RIA).

(RIA) *implies* (MGA.). Trivial.

(MGA) *implies* (YIA). (see [6]) Assume that $n \geq 2$. Let $k \in \{1, 2, \ldots, n-1\}$. For $a_1 = a_2 = \ldots = a_k = a$ and $a_{k+1} = a_{k+2} = \ldots = a_n = b$ the Cauchy mean inequality (MGA) becomes

$$(a^k b^{n-k})^{1/n} \leq \frac{ka + (n-k)b}{n}$$

or equivalent

$$a^{k/n} b^{1-k/n} \leq \frac{k}{n} a + \left(1 - \frac{k}{n}\right) b.$$

Then for all $d \in (0,1) \cap \mathbb{Q}$ we have $a^d b^{1-d} \leq da + (1-d)b$. If $\beta \in (0,1)$ then there exists a rational sequence $(d_j)_j \subset (0,1) \cap \mathbb{Q}$ convergent to β. It follows that $a^\beta b^{1-\beta} = \lim_{j \to +\infty} a^{d_j} b^{1-d_j} \leq \lim_{j \to +\infty} (d_j a + (1-d_j)b) = \beta a + (1-\beta)b$.

Lemma 16. *The Young inequality* (YIC)*, the Rogers-Hölder inequality* (RHA)*, the Cauchy-Bunyakovski-Schwarz inequality* (CBS)*, the root mean square inequality* (RMS)*, the Euclid ordinary inequality* (IOE)*, the Cauchy mean inequality* (MGA) *and the Young inequality* (YIA) *are equivalent.*

Proof. (YIC) *implies* (RHA). Let $k \in \{1, 2, \ldots, n\}$. Put

$$A = \left(\sum_{j=1}^n a_j^p\right)^{1/p}, \quad B = \left(\sum_{j=1}^n b_j^q\right)^{1/q}$$

and $u_k = a_k/A$, $v_k = b_k/B$. Using the Young inequality (YIC), we have

$$\frac{a_k}{A}\frac{b_k}{B} = u_k v_k \leq \frac{u_k^p}{p} + \frac{v_k^q}{q} = \frac{a_k^p}{pA^p} + \frac{b_k^q}{qB^q}$$

and summing over k from 1 to n, we obtain

$$\frac{1}{AB}\sum_{k=1}^{n} a_k b_k \leq 1$$

which is the Rogers-Hölder inequality (RHA).

(RHA) *implies* (CBS). The Cauchy-Bunyakovski-Schwarz inequality (CBS) is a special case of the Rogers-Hölder inequality (RHA).

(CBS) *implies* (RMS). For $b_1 = b_2 = \ldots = b_n = 1$ the Cauchy-Bunyakovski-Schwarz inequality (CBS) is exactly the root mean square inequality (RMS).

(RMS) *implies* (IOE). We observe that $\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$ is equivalent to $(a-b)^2 \geq 0$. On the other hand the Cauchy ordinary inequality (IOE), $ab \leq ((a+b)/2)^2$ is equivalent to $(a-b)^2 \geq 0$. Finally, note that, the assertions, the Euclid ordinary inequality (IOE) and the Cauchy mean inequality (MGA) are equivalent, the Cauchy mean inequality (MGA) implies the Young inequality (YIA) and the Young inequality (YIA) and the Young inequality (YIC) are equivalent, are proved.

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