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ON THE LAPLACE OPERATOR OF A TUBE SURFACE IN EUCLIDEAN SPACE

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Abstract. In this paper, we study the Euclidean version of tube surfaces in 3-dimensional Euclidean space $\mathbf{E}^3$ and characterize it by its Gauss map. Moreover, under the condition $\Delta^I \overrightarrow{G} = \lambda \overrightarrow{G}$ where Δ^I denotes the Laplace operator with respect to the first fundamental form I, $\lambda \in \mathbb{R}$ we investigate some special quantities with respect to the Gaussian and mean curvatures. Furthermore, some important theorems are obtained for that one and we have shown that the tube surface under study is developable and not minimal. Finally, examples of tube surfaces are used to demonstrate our theoretical results and plotted.

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1. Introduction

It is well known that, the theory of Gauss map G is always one of interesting topics in a Euclidean space and it has been investigated from the various viewpoints by many differential geometers. Let $Y : M \to \mathbf{E}^3$ be an isometric immersion of a surface in Euclidean 3-space. Denotes by G and Δ, respectively, the Gauss map and the Laplacian operator of the surface M with respect to the induced metric form that of $\mathbf{E}^3$. Takahashi [1] proved that the minimal surfaces and the spheres are the only surfaces in $\mathbf{E}^3$ satisfying the condition

$$\Delta Y = AY \tag{1}$$

where $A \in \mathbb{R}^{3\times 3}$ the set of 3×3 real matrices. Garay [2] extended it to the hypersurfaces, that is, he studied the hypersurfaces in $\mathbf{E}^{n+1}$. On the other hand, Baikoussis and Blair [3] studied ruled surfaces such that their Gauss maps satisfy

$$\Delta G = AG. \tag{2}$$

They showed that the only ones are planes and circular cylinders. Also, for the Lorentz version S. M. Choi [4] showed that the only ruled surfaces with non-null base curve in a 3-dimensional Minkowski space $\mathbf{E}_1^3$ satisfying the condition (2) are locally the Euclidean plane, the Minkowski space, the Lorentz hyperbolic cylinder, the Lorentz circular cylinder and the hyperbolic cylinder. L. J. Alias, A. Ferrandez, P. Lucas and M. A. Merono [5] proved that the only ruled surfaces in $\mathbf{E}_1^3$ with null rulings satisfying the condition (2) are B-scrolls over null curves.

Moreover, on the generalization of equation $\Delta G = \lambda G,\ \lambda \in \mathbb{R}$, surfaces whose Gauss map is an eigenfunction of a Laplacian, Dillen F., Pas J., and Verstraelen L. [6] studied surfaces of revolution in a Euclidean 3-space $\mathbf{E}^3$ such that its Gauss map G satisfies the condition 2. Also, D. W. Yoon [7] investigated a non-developable ruled surface in a Euclidean 3-space whose mean curvature vector is an eigenvector of the Laplacian operator with respect to non-degenerate second fundamental form of the surface.

In this paper, motivated by the results given in [7], we investigate a tube surface in Euclidean 3-space $\mathbf{E}^3$ satisfying the following conditions:

$$\Delta^I \overrightarrow{K} = \lambda \overrightarrow{K}, \quad \lambda \in \mathbb{R}, \tag{3}$$

$$\Delta^I \overrightarrow{H} = \lambda \overrightarrow{H}, \tag{4}$$

$$\Delta^I \overrightarrow{\Gamma} = \lambda \overrightarrow{\Gamma}, \quad \Gamma = \frac{K}{H}, \tag{5}$$

$$\Delta^I \overrightarrow{\Omega} = \lambda \overrightarrow{\Omega}, \quad \Omega = KH, \tag{6}$$

$$\Delta^I \overrightarrow{\Pi} = \lambda \overrightarrow{\Pi}, \quad \Pi = aK + bH \ \text{ and } a,\ b \in \mathbb{R}, \tag{7}$$

2. Preliminaries

In Euclidean 3-space $\mathbf{E}^3$, it is well known that to each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields $e_1,\ e_2$ and e_3 are respectively, the tangent, the principal normal and the binormal vector fields [8]. We consider the usual metric in Euclidean 3-space $\mathbf{E}^3$, that is,

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. In particular, the norm of a vector $X \in \mathbf{E}^3$ is given by

$$\|X\| = \sqrt{\langle X, X\rangle}.$$

If $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ are arbitrary vectors in $\mathbf{E}^3$, we define the vector product of X and Y as the following:

$$X \wedge Y = (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1). \tag{8}$$

Let $\delta : I \subset \mathbb{R} \to \mathbf{E}^3,\ \delta = \delta(s)$, be an arbitrary curve in $\mathbf{E}^3$. The curve δ is said to be of unit speed (or parameterized by the arc-length parameter s) if $\langle \delta'(s), \delta'(s) \rangle = 1$ for any $s \in I$. Let $\{e_1(s),\ e_2(s),\ e_3(s)\}$ be the moving frame of δ, where the vectors $e_1,\ e_2$ and e_3 are mutually orthogonal vectors. A frame $\{e_1,\ e_2,\ e_3\}$ satisfies the following so called the Frenet equations [9]:

$$\begin{bmatrix} e_1'(s) \\ e_2'(s) \\ e_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{bmatrix}, \tag{9}$$

where

$$\begin{cases} \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1, \\ \langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_1 \rangle = 0, \\ \det(e_1, e_2, e_3) = 1. \end{cases}$$

Let $M : \Phi = \Phi(s,t) \subset \mathbf{E}^3$ be a regular surface. Then the unit normal vector field of a surface M is defined by

$$U = \frac{\Phi_s \wedge \Phi_t}{\|\Phi_s \wedge \Phi_t\|}, \tag{10}$$

where $\Phi_s = \frac{\partial \Phi(s,t)}{\partial s}$. For the components g^{ij} ($i,\ j$ belong to $\{1,2\}$) of the induced metric $\langle .,. \rangle$ on M from that of $\mathbf{E}^3$ we denote by $\{g^{ij}\}$ (resp. D) the inverse matrix (resp. the determinant) of the matrix g_{ij}. Then, the Laplacian Δ on M is given by [10]. If $\Phi : M \to \mathbb{R},\ (s,t) \to \Phi(s,t)$ is a smooth function and Δ^I the Laplace operator with respect to the first I fundamental form of M, then from [10],we have

$$\begin{aligned} I &= g_{11} ds^2 + 2 g_{12} ds dt + g_{22} dt^2, \\ g_{11} &= \langle \Phi_s, \Phi_s \rangle,\ g_{12} = \langle \Phi_s, \Phi_t \rangle,\ g_{22} = \langle \Phi_t, \Phi_t \rangle, \end{aligned}$$

and

$$\Delta^I = \frac{1}{\sqrt{|\ D\ |}} \sum_{i,j} \frac{\partial}{\partial u^i} \Big(\sqrt{|\ D\ |} g^{ij} \frac{\partial}{\partial u^j} \Big). \tag{11}$$

Also, the second fundamental form of the surface M is given by

$$II = -\langle dU, d\Phi \rangle = h_{11} ds^2 + 2 h_{12} ds dt + h_{22} dt^2, \tag{12}$$

$$h_{11} = \langle \Phi_{ss}, U \rangle,\ h_{12} = \langle \Phi_{st}, U \rangle,\ h_{22} = \langle \Phi tt, U \rangle. \tag{13}$$

The Gaussian and mean curvature of the surface M in $\mathbf{E}^3$, respectively, are [10]

$$K(s,t) = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}, \tag{14}$$

$$H(s,t) = \frac{g_{11}h_{11} - 2g_{12}h_{12} + g_{22}h_{22}}{2\left(g_{11}g_{22} - g_{12}^2\right)}, \tag{15}$$

where h_{ij} are the components of second fundamental form.

3. Tube surfaces in $\mathbf{E}^3$

Let $(\alpha, F) = (\alpha(s), F(s))$ be a unit speed curve with frame field $F = \{e_1,\ e_2,\ e_3\}$. A tube surface is a surface which has a parametrization in the following form:

$$M:\ \Phi(s,t) = \alpha(s) + r\Big(\cos[t]e_2(s) + \sin[t]e_3(s)\Big), \tag{16}$$

Calculating the partial derivative of (16) with respect to s and t respectively, we get

$$\begin{aligned}\Phi_s &= Q\ e_1 - r\tau\Big[\sin[t]e_2 - \cos[t]e_3\Big],\\ \Phi_t &= -r\Big[\sin[t]e_2 - \cos[t]e_3\Big],\end{aligned}$$

where $Q = 1 - r\kappa\cos[t]$. From which, the components of the first fundamental form are

$$g_{11} = Q^2 + r^2\tau^2, \quad g_{12} = r^2\tau, \quad g_{22} = r^2. \tag{17}$$

The unit normal vector on Φ can be directly obtained from (10) getting

$$U = -\cos[t]e_2 - \sin[t]e_3. \tag{18}$$

Then, the components of the second fundamental form of Φ are obtained by

$$h_{11} = r\tau^2 - \kappa Q\cos[t], \quad h_{12} = r\tau, \quad h_{22} = r. \tag{19}$$

The matrix (g^{ij}) reads as follows:

$$(g^{ij}) = \frac{1}{r^2Q^2}\begin{bmatrix} r^2 & -r^2\tau \\ -r^2\tau & Q^2 + r^2\tau^2 \end{bmatrix}.$$

Based on the above calculations, the Gaussian curvature K and the mean curvature H of (16) are given by

$$K = -\frac{\kappa\cos[t]}{rQ}, \tag{20}$$

$$H = \frac{1 - 2r\kappa\cos[t]}{2rQ}. \tag{21}$$

Thus, using (11) we show that the Laplacian Δ^I of M can be expressed as:

$$\Delta^I = \tfrac{r}{Q^3}\Big(\kappa\tau\sin[t] + \kappa'\cos[t]\Big)\tfrac{\partial}{\partial s} + \tfrac{1}{4rQ^3}\Big(- 4r\tau' + 4r^2(\kappa\tau' - \kappa'\tau)\cos[t] + \kappa[4 + r^2(\kappa^2 - 4\tau^2)]\sin[t] - 4r\kappa^2\sin[2t] + r^2\kappa^3\sin[3t]\Big)\tfrac{\partial}{\partial t} - \Big(\tfrac{2\tau}{Q^2}\Big)\tfrac{\partial^2}{\partial s\partial t} + \Big(\tfrac{1}{Q^2}\Big)\tfrac{\partial^2}{\partial s^2} + \Big(\tfrac{1}{r^2} + \tfrac{\tau^2}{Q^2}\Big)\tfrac{\partial^2}{\partial t^2}. \tag{22}$$

4. Main results

Now, we shall give a detailed discussion on a tube surface in $\mathbf{E}^3$

4.1. Laplacian of the Gaussian curvature

From the equations (3), (18), (20) and (22), the Laplacian of $\overrightarrow{K}$ on the surface (16) is given by:

$$\tfrac{1}{16r^3Q^5}\Big(0, A_0 + \sum_{i=1}^{6} A_i\cos[it] + \sum_{j=1}^{4} B_j\sin[jt], C_0 + \sum_{i=1}^{4} C_i\cos[it] + \sum_{j=1}^{6} D_j\sin[jt]\Big) = \tfrac{\lambda\kappa\cos[t]}{rQ}\Big(0, \cos[t], \sin[t]\Big), \tag{23}$$

which implies

$$A_0 + \sum_{i=1}^{6} A_i\cos[it] + \sum_{j=1}^{4} B_j\sin[jt] = 16\lambda r^2 Q^4\kappa\cos^2[t], \tag{24}$$

and

$$C_0 + \sum_{i=1}^{4} C_i\cos[it] + \sum_{j=1}^{6} D_j\sin[jt] = 16\lambda r^2 Q^4\kappa\cos[t]\sin[t], \tag{25}$$

where

$$A_0 = -4r^2\Big[\kappa^3\Big[10 + r^2(\kappa^2 + 2\tau^2)\Big] - 2\kappa''\Big],$$

$$A_1 = 12r^3(3\kappa'^2 - \kappa\kappa'') + 2r\kappa^2\Big[21r^2\kappa^2 + 2(15 + 11r^2\tau^2)\Big],$$

$$A_2 = 8(r^2\kappa'' - 4\kappa) - r^2\kappa\Big[\kappa^2(72 + 7r^2\kappa^2) + 8\tau^2(4 + r^2\kappa^2)\Big],$$

$A_3 = 4r^3(3\kappa'^2 - \kappa\kappa'') + r\kappa^2\Big[29r^2\kappa^2 + 4(13 + r^2\tau^2)\Big],$
$A_4 = -4r^2\kappa^3\Big[8 + r^2\kappa^2\Big], \quad A_5 = 9r^3\kappa^4, \quad A_6 = -r^4\kappa^5,$
$B_1 = -12r^3\kappa\Big[\kappa\tau' - \kappa'\tau\Big], \quad B_2 = 4r^2\Big[\kappa\tau'(4 + r^2\kappa^2) + \kappa'\tau(8 - r^2\kappa^2)\Big],$
$B_3 = -12r^3\kappa\Big[\kappa\tau' - \kappa'\tau\Big], \quad B_4 = 2r^4\kappa^2\Big[\kappa\tau' - \kappa'\tau\Big],$
$C_0 = -6r^4\kappa^2\Big[\kappa\tau' - \kappa'\tau\Big], \quad C_1 = 4r^3\kappa\Big[5\kappa\tau' + 7\kappa'\tau\Big],$
$C_2 = -8r^2\Big[\kappa\tau'(2 + r^2\kappa^2) + \kappa'\tau(4 - r^2\kappa^2)\Big], \quad C_3 = 12r^3\kappa\Big[\kappa\tau' - \kappa'\tau\Big],$
$C_4 = -2r^4\kappa^2\Big[\kappa\tau' - \kappa'\tau\Big]$
and
$D_1 = 4r^3(3\kappa'^2 - \kappa\kappa'') + 2r\kappa^2\Big[11r^2\kappa^2 + 4(17 + 13r^2\tau^2)\Big],$
$D_2 = 8r^2\kappa'' - \kappa\Big[5r^2\kappa^2(16 + r^2\kappa^2) + 32(1 + r^2\tau^2)\Big],$
$D_3 = 4r^3(3\kappa'^2 - \kappa\kappa'') + r\kappa^2\Big[31r^2\kappa^2 + 4(13 + r^2\tau^2)\Big],$
$D_4 = -4r^2\kappa^3\Big[8 + r^2\kappa^2\Big], \quad D_5 = 9r^3\kappa^4, \quad D_6 = -r^4\kappa^5.$

Now, according to the angle t we have two cases to be discussed as follows:

Case 1.

If we take the angle $t = \frac{\pi}{2}\Big(2n + 1\Big),\ n = 0,\ \pm 2,\ \pm 4, ...$, then from (24) and (25) we have, respectively

$$32\kappa\Big[1 + r^2\tau^2\Big] = 0, \tag{26}$$

$$16r\Big[\kappa^2(1 + 3r^2\tau^2) + r(\kappa\tau' + 2\kappa'\tau)\Big] = 0. \tag{27}$$

It refers to the Gaussian curvature K vanishes identically. On the other hand, if $\kappa = 0$. Thus M is an open part of a circular cylinder.

Case 2.

The same is hold when $t = \frac{\pi}{2}\Big(2n + 1\Big),\ n = \pm 1,\ \pm 3,\ \pm 5, ...$

From this, our major result states as follows.

Theorem 1. *Let M be a tube surface given by (16) in $\boldsymbol{E}^3$. Then, the following are equivalent:*

(1): *M is a developable surface or an open part of a circular cylinder.*

(2): *M satisfies the equation $\Delta^I \vec{K} = \lambda \vec{K},\ \lambda \in \mathbb{R}$.*

4.2. Laplacian of the mean curvature

From the equations (4), (18) and (21), the Laplacian Δ^I of $\overrightarrow{H}$ on M is expressed as

$$\frac{-1}{16r^3Q^5}\Big(0, A_0+\sum_{i=1}^{6}A_i\cos[it]+\sum_{j=1}^{4}B_j\sin[jt], C_0+\sum_{i=1}^{4}C_i\cos[it]+\sum_{j=1}^{6}D_j\sin[jt]\Big)$$

$$=-\frac{\lambda\Big(1-2r\kappa\cos[t]\Big)}{2rQ}\Big(0,\cos[t],\sin[t]\Big), \tag{28}$$

which leads to

$$A_0+\sum_{i=1}^{6}A_i\cos[it]+\sum_{j=1}^{4}B_j\sin[jt]=8\lambda r^2Q^4\cos[t]\big(1-2r\kappa\cos[t]\big), \tag{29}$$

$$C_0+\sum_{i=1}^{4}C_i\cos[it]+\sum_{j=1}^{6}D_j\sin[jt]=8\lambda r^2Q^4\sin[t]\big(1-2r\kappa\cos[t]\big), \tag{30}$$

with the notion

$A_0=4r\Big[\kappa[4+r^2(4\tau^2+\kappa^2(11+r^2(\kappa^2+2\tau^2)))]-r^2\kappa''\Big]$,

$A_1=-2\Big[4+r^2[4\tau^2+\kappa^2[41+r^2(21\kappa^2+22\tau^2)]+3r^2(3\kappa'^2-\kappa\kappa'')]\Big]$,

$A_2=r\Big[\kappa\Big[40+r^2[24\tau^2+\kappa^2(76+r^2(7\kappa^2+8\tau^2))]\Big]-4r^2\kappa''\Big]$,

$A_3=-r^2\Big[\kappa^2[54+r^2(29\kappa^2+4\tau^2)]+2r^2(3\kappa'^2-\kappa\kappa'')\Big]$,

$A_4=4r^3\kappa^3\Big[8+r^2\kappa^2\Big]$, $\quad A_5=-9r^4\kappa^4$, $\quad A_6=r^5\kappa^5$,

$B_1=2r^2\Big[4\tau'+r^2\kappa(6\kappa\tau'-5\kappa'\tau)\Big]$,

$B_2=-4r^3\Big[\kappa\tau'(5+r^2\kappa^2)+\kappa'\tau(3-r^2\kappa^2)\Big]$,

$B_3=2r^4\kappa\Big[6\kappa\tau'-5\kappa'\tau\Big]$, $\quad B_4=-2r^5\kappa^2\Big[\kappa\tau'-\kappa'\tau\Big]$,

$C_0=2r^3\Big[2(3\kappa\tau'-\kappa'\tau)+3r^2\kappa^2(\kappa\tau'-\kappa'\tau)\Big]$,

$C_1=-2r^2\Big[4\tau'+r^2\kappa(14\kappa\tau'+\kappa'\tau)\Big]$,

$C_2=4r^3\Big[(5\kappa\tau'+3\kappa'\tau)+2r^2\kappa^2(\kappa\tau'-\kappa'\tau)\Big]$,

$C_3=-2r^4\kappa\Big[6\kappa\tau'-5\kappa'\tau\Big]$, $\quad C_4=2r^5\kappa^2\Big[\kappa\tau'-\kappa'\tau\Big]$,

and

$D_1=-2\Big[4+r^2[4\tau^2+\kappa^2(31+2r^2(5\kappa^2+7\tau^2))+r^2(3\kappa'^2-\kappa\kappa'')]\Big]$,

$D_2=r\Big[8\kappa(5+3r^2\tau^2)+r^2[\kappa^3(72+5r^2\kappa^2)-4\kappa'']\Big]$,

$D_3 = -r^2\Big[\kappa^2[54 + r^2(29\kappa^2 + 4\tau^2)] + 2r^2(3\kappa'^2 - \kappa\kappa'')\Big]$,
$D_4 = 4r^3\kappa^3\Big[8 + r^2\kappa^2\Big]$, $\quad D_5 = -9r^4\kappa^4$, $\quad D_6 = r^5\kappa^5$.
Consider now the following two cases:
Case 1.
If $t = \frac{\pi}{2}\Big(2n+1\Big)$, $n = 0, \pm 2, \pm 4, ...$, we have from the equations (29) and (30)

$$r\Big[\kappa(3 + r^2\tau^2) - r\tau'\Big] = 0, \tag{31}$$

$$16r\Big[1 + r^2[\kappa^2 + \tau^2(1 + 3r^2\kappa^2)] + r^3(\kappa\tau' + 2\kappa'\tau)\Big] = \lambda. \tag{32}$$

Case 2.
When $t = \frac{\pi}{2}\Big(2n+1\Big)$, $n = \pm 1, \pm 3, \pm 5, ...$, similar result can be obtained.

Thus the tube surface (16) has constant mean curvature $H = \frac{1}{2r}$ and we summarize the following theorem:

Theorem 2. *Let M be a tube surface given by (16) in $\boldsymbol{E}^3$. Then, the following conditions are equivalent:*
(1): *M has non-zero mean curvature (it is not a minimal surface).*
(2): *M satisfies the equation $\Delta^I \vec{H} = \lambda\vec{H}$, $\lambda \in \mathbb{R}$.*

4.3. Laplacian of the form $\boldsymbol{\Gamma} = \frac{K}{H}$

Now, by using (18), (20) and (21), eq. (5) can be expressed as

$$\frac{1}{4r^2Q^3\Big(1-2r\kappa\cos[t]\Big)^3}\Big(0,\ A_0 + \sum_{i=1}^{7} A_i\cos[it] + \sum_{j=1}^{5} B_j\sin[jt], C_0 + \sum_{i=1}^{5} C_i\cos[it] + \sum_{j=1}^{7} D_j\sin[jt]\Big)$$
$$= \frac{\lambda\kappa}{1-2r\kappa\cos[t]}\Big(0,\ 2\cos^2[t],\ \sin[2t]\Big), \tag{33}$$

where the coefficients $A_0, A_1, ..., B_1, ..., C_1, ..., D_1, .., D_5$ are as follows:
$A_0 = -2r^2\Big[\kappa^3\Big[19r^2\kappa^2 + 2(17 + 9r^2\tau^2)\Big] + 3r^2\kappa(3\kappa'^2 - \kappa\kappa'') - 2\kappa''\Big]$,
$A_1 = r\Big[\kappa^2\Big[8(7 + 6r^2\tau^2) + r^2\kappa^2[15r^2\kappa^2 + 4(35 + 6r^2\tau^2)]\Big] + 6r^2(5\kappa'^2 - 3\kappa\kappa'')\Big]$,
$A_2 = 4r^2\kappa''(1 + 2r^2\kappa^2) - 8\kappa\Big[2 + r^2\Big[2\tau^2 + \kappa^2[14 + r^2(8\kappa^2 + 5\tau^2)] + 3r^2\kappa'^2\Big]\Big]$,
$A_3 = r\Big[\kappa^2\Big[8(5 + 2r^2\tau^2) + r^2\kappa^2[11r^2\kappa^2 + 2(45 + 4r^2\tau^2)]\Big] + 2r^2(5\kappa'^2 - 3\kappa\kappa'')\Big]$,
$A_4 = -2r^2\kappa\Big[\kappa^2\Big[2(11 + r^2\tau^2) + 17r^2\kappa^2\Big] + r^2(3\kappa'^2 - \kappa\kappa'')\Big]$,
$A_5 = r^3\kappa^4\Big[26 + 5r^2\kappa^2\Big]$, $\quad A_6 = -8r^4\kappa^5$, $\quad A_7 = r^5\kappa^6$,
$B_1 = -2r^3\kappa\Big[2\kappa\tau'(4 + r^2\kappa^2) + \kappa'\tau(1 - 2r^2\kappa^2)\Big]$,

$B_2 = 4r^2\Big[\kappa\tau'(2+5r^2\kappa^2) + 4\kappa'\tau(1-r^2\kappa^2)\Big]$,
$B_3 = -2r^3\kappa\Big[\kappa\tau'(8+3r^2\kappa^2) + \kappa'\tau(1-3r^2\kappa^2)\Big]$, $B_4 = 2r^4\kappa^2\Big[5\kappa\tau' - 4\kappa'\tau\Big]$,
$B_5 = -2r^5\kappa^3\Big[\kappa\tau' - \kappa'\tau\Big]$, $C_0 = -2r^4\kappa^2\Big[11\kappa\tau' + 4\kappa'\tau\Big]$,
$C_1 = 2r^3\kappa\Big[2\kappa\tau'(6+5r^2\kappa^2) + \kappa'\tau(19-10r^2\kappa^2)\Big]$, $C_2 = -8r^2\Big[\kappa\tau'(1+4r^2\kappa^2) + 2\kappa'\tau\Big]$,
$C_3 = 2r^3\kappa\Big[\kappa\tau'(8+5r^2\kappa^2) + \kappa'\tau(1-5r^2\kappa^2)\Big]$, $C_4 = -2r^4\kappa^2\Big[5\kappa\tau' - 4\kappa'\tau\Big]$,
$C_5 = 2r^5\kappa^3\Big[\kappa\tau' - \kappa'\tau\Big]$,
and
$D_1 = r\Big[\kappa^2\Big[8(8+7r^2\tau^2) + r^2\kappa^2(66+5r^2\tau^2)\Big] + 2r^2(5\kappa'^2 - 3\kappa\kappa'')\Big]$,
$D_2 = -4\Big[\kappa\Big[4 + r^2[4\tau^2 + 3\kappa'^2 + \kappa^2[30 + r^2(11\kappa^2 + 8\tau^2)]\Big] - r^2\kappa''(1+r^2\kappa^2)\Big]$,
$D_3 = r\Big[\kappa^2\Big[8(5+2r^2\tau^2) + r^2\kappa^2(92+9r^2\kappa^2)\Big] + 2r^2(5\kappa'^2 - 3\kappa\kappa'')\Big]$,
$D_4 = -2r^2\kappa\Big[\kappa^2\Big[2(11+r^2\tau^2) + 17r^2\kappa^2\Big] + r^2(3\kappa'^2 - \kappa\kappa'')\Big]$,
$D_5 = r^3\kappa^4\Big[26 + 5r^2\kappa^2\Big]$, $D_6 = -8r^4\kappa^5$, $D_7 = r^5\kappa^6$.

From (33), when $t = \frac{\pi}{2}\Big(2n+1\Big)$, $n \in Z$, we get

$$\kappa(1 + r^2\tau^2) = 0, \tag{34}$$

$$\pm\,\kappa^2(3 + 5r^2\tau^2) + r(\kappa\tau' + 2\kappa'\tau) = 0. \tag{35}$$

From (20), (21), (34) and (35), we have $K = 0$ and $H = \frac{1}{2r}$.

4.4. Laplacian of the form $\Omega = \mathbf{KH}$

As in the above case, by adopting (18), (20) and (21), eq. (6) can be written as

$$\frac{1}{32r^4Q^6}\Big(0,\ A_0 + \sum_{i=1}^{7} A_i\cos[it] + \sum_{j=1}^{5} B_j\sin[jt], C_0 + \sum_{i=1}^{5} C_i\cos[it] + \sum_{j=1}^{7} D_j\sin[jt]\Big)$$
$$= \frac{\lambda\kappa\Big(1-2r\kappa\cos[t]\Big)}{4r^2Q^2}\Big(0,\ 2\cos^2[t],\ \sin[2t]\Big). \tag{36}$$

Similarly, straightforward computations at $t = \frac{\pi}{2}\Big(2n+1\Big)$, $n = 0, \pm 2, \pm 4, ..$and $t = \frac{\pi}{2}\Big(2n+1\Big)$, $n = \pm 1, \pm 3, \pm 5, ..$, lead to

$$K = 0 \text{ and } H = \frac{1}{2r}.$$

4.5. Laplacian of the form Π= aK + bH

Parallel to the study that considered in the previous cases, eq.(7) can be given as

$$\frac{-1}{16r^3Q^5}\Big(0,\ A_0+\sum_{i=1}^{6}A_i\cos[it]+\sum_{j=1}^{4}B_j\sin[jt], C_0+\sum_{i=1}^{4}C_i\cos[it]+\sum_{j=1}^{6}D_j\sin[jt]\Big)$$
$$=\frac{-\lambda\Big(b-2(a+br)\kappa\cos[t]\Big)}{2rQ}\Big(0,\ \cos[t],\ \sin[t]\Big). \tag{37}$$

which leads to

$$A_0+\sum_{i=1}^{6}A_i\cos[it]+\sum_{j=1}^{4}B_j\sin[jt]=8\lambda r^2Q^4\cos[t]\big(b-2(a+br)\kappa\cos[t]\big),$$

$$C_0+\sum_{i=1}^{4}C_i\cos[it]+\sum_{j=1}^{6}D_j\sin[jt]=8\lambda r^2Q^4\sin[t]\big(b-2(a+br)\kappa\cos[t]\big),$$

Here, when $t=\frac{\pi}{2}\Big(2n+1\Big),\ n\in Z$, Eq. (37) gives two equations

$$\mp br^2\tau'+\kappa\Big[4a+3br+r^2\tau^2(4a+br)\Big]=0,$$

and

$$16r\Big[b+r\Big[\kappa^2(2a+br)(1+3r^2\tau^2)+r\tau[b\tau\pm 2\kappa'(2a+br)]\pm r\kappa\tau'(2a+br)\Big]\Big]=\lambda b.$$

From which

$$K=0 \text{ and } H=\frac{1}{2r}.$$

Based on the above discussions, we state the following theorem

Theorem 3. *Let M be a tube surface as given in (16), then Eqs. (5), (6) and (7) are satisfied if and only if the surface (16) is developable and not minimal.*

We now present some of very typical examples.

5. Examples

Example 1. Let $\alpha(s)$ be a *circular helix* in $\mathbf{E}^3$ with Frenet frame $F=\{e_1,\ e_2,\ e_3\}$ as follows:

$$\alpha(s)=\Big(a\cos[s],\ a\sin[s],\ bs\Big),\quad a,b\in\mathbb{R} \text{ and } a>0, \tag{38}$$

$$\begin{aligned}
\mathbf{e}_1(s) &= \tfrac{1}{\sqrt{a^2+b^2}}\Big(-a\sin[s],\ a\cos[s],\ b\Big),\\
\mathbf{e}_2(s) &= -\Big(\cos[s],\ \sin[s],\ 0\Big),\\
\mathbf{e}_3(s) &= \tfrac{1}{\sqrt{a^2+b^2}}\Big(b\sin[s],\ -b\cos[s],\ a\Big).
\end{aligned}\tag{39}$$

In the light of the above, the curvature and the torsion of the curve are respectively, given by

$$\kappa = \frac{a}{a^2+b^2},\quad \tau = \frac{b}{a^2+b^2}.$$

The map $\Phi = \big(\Phi_1, \Phi_2, \Phi_3\big) : \mathbb{R}^2 \to \mathbf{E}^3$ defines a tube surface M in $\mathbf{E}^3$ where:

$$\begin{aligned}
\Phi_1 &= \big(a - r\cos[t]\big)\cos[s] + \tfrac{b\ r}{\sqrt{a^2+b^2}}\sin[s]\sin[t],\\
\Phi_2 &= \big(a - r\cos[t]\big)\sin[s] - \tfrac{b\ r}{\sqrt{a^2+b^2}}\cos[s]\sin[t],\\
\Phi_3 &= b\ s + \tfrac{a\ r}{\sqrt{a^2+b^2}}\sin[t].
\end{aligned}\tag{40}$$

For this surface, the Gaussian curvature K and the mean curvature H are defined by

$$K = -\frac{a\cos[t]}{r\big(a^2+b^2-a\ r\cos[t]\big)},\tag{41}$$

$$H = \frac{a^2+b^2-2a\ r\cos[t]}{2r\big(a^2+b^2-a\ r\cos[t]\big)}.\tag{42}$$

When $t = \frac{\pi}{2}\Big(2n+1\Big),\ n \in Z$, Eqs. (41) and (42) lead to

$$K = 0,\quad H = \frac{1}{2r},$$

i.e., the surface (40) is a developable and not minimal.
It seems natural to see that the tube surface (40) satisfies the Eqs. (3)-(7) as shown in Figure 1.

Example 2. Let $\beta : I \subset \mathbb{R} \to \mathbf{E}^3$ be a *circular helix* in $\mathbf{E}^3$ and consider the tube surface parameterized by

$$\Psi(s,t) = \beta(s) + r\chi,\tag{43}$$

where

$$\beta(s) = \Big(\cos[\frac{s}{3}] - 1,\ \sin[\frac{s}{3}],\ \frac{2\sqrt{2}}{3}s\Big),\tag{44}$$

$$\chi = -\Big(-\cos[\frac{s}{3}]\cos[t] + \frac{2\sqrt{2}}{3}\sin[\frac{s}{3}]\sin[t], -\sin[\frac{s}{3}]\cos[t] - \frac{2\sqrt{2}}{3}\cos[\frac{s}{3}]\sin[t], \frac{1}{3}\sin[t]\Big).\tag{45}$$

It is easy to calculate the Frenet frame as follows

$$\begin{aligned}\mathbf{e}_1(s) &= \tfrac{1}{3}\Big(-\sin[\tfrac{s}{3}], \cos[\tfrac{s}{3}], 2\sqrt{2}\Big),\\ \mathbf{e}_2(s) &= -\Big(\cos[\tfrac{s}{3}],\ \sin[\tfrac{s}{3}], 0\Big),\\ \mathbf{e}_3(s) &= \tfrac{1}{3}\Big(2\sqrt{2}\sin[\tfrac{s}{3}], -2\sqrt{2}\cos[\tfrac{s}{3}], 1\Big).\end{aligned}$$

The curvature κ and the torsion τ of β are given by $\kappa = \frac{1}{9}$ and $\tau = \frac{2\sqrt{2}}{9}$
Let us write down, as usually,

$$\begin{aligned}\Psi_s &= \tfrac{\partial\Psi}{\partial s} = \tfrac{1}{3}\Big(-(1-r\cos[t])\sin[\tfrac{s}{3}] + \tfrac{2\sqrt{2}}{3}r\cos[\tfrac{s}{3}]\sin[t], (1-r\cos[t])\cos[\tfrac{s}{3}] + \tfrac{2\sqrt{2}}{3}\sin[\tfrac{s}{3}]\sin[t], 2\sqrt{2}\Big),\\ \Psi_t &= \tfrac{\partial\Psi}{\partial t} = \tfrac{r}{3}\Big(2\sqrt{2}\sin[\tfrac{s}{3}]\cos[t] + 3\cos[\tfrac{s}{3}]\sin[t], -2\sqrt{2}\cos[\tfrac{s}{3}]\cos[t] + 3\sin[\tfrac{s}{3}]\sin[t], \cos[t]\Big).\end{aligned} \tag{46}$$

A straight forward computation leads to the Gaussian curvature K and the mean curvature H in the following forms

$$K = \frac{\cos[t]}{r(-9 + r\cos[t])} \tag{47}$$

$$H = \frac{9 - 2r\cos[t]}{-18r + 2r^2\cos[t]} \tag{48}$$

From aforementioned data, one can deduce that the Laplace operator on Ψ corresponding to the induced metric form satisfies the Eqs. (3)-(7). One can see the graph of Ψ in Figure 2.

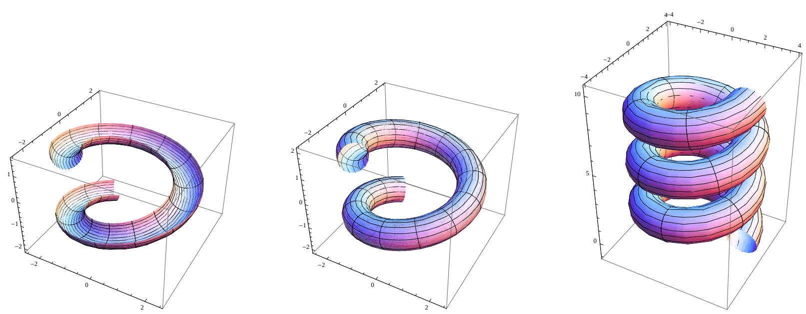

Figure 1: From the left to the right, some tube surfaces generated by the circular helix α are shown as follows: $r = 0.5,\ a = 2,\ b = 0.4,\ s \in [-4, 4],\ t \in [\pi, 2\pi]$; $r = 0.5,\ a = 2,\ b = 0.4,\ s \in [-4, 4],\ t \in [-\pi, \pi]$; $r = 0.5,\ a = 2,\ b = 0.4,\ s \in [0, 6\pi],\ t \in [0, 6\pi]$.

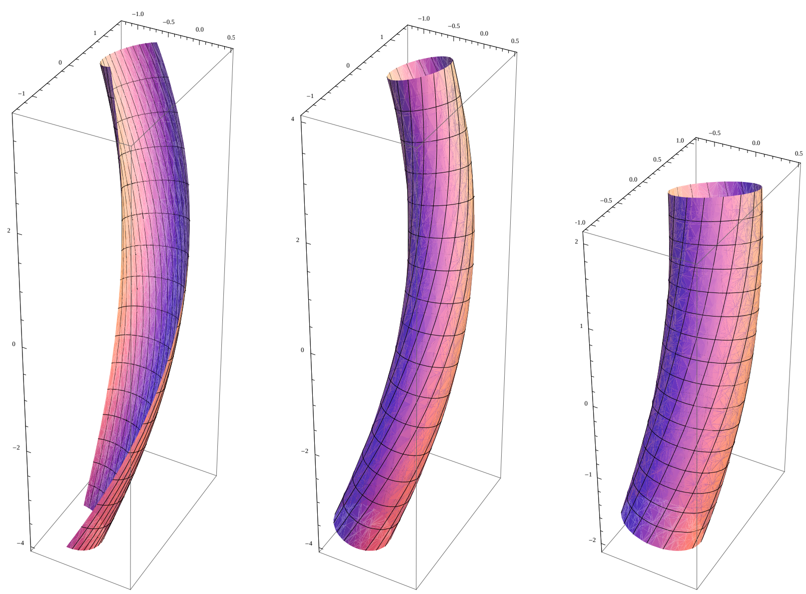

Figure 2: From the left to the right, some tube surfaces generated by the *circular helix* β can be seen as: $r = 0.5,\ s \in [-4, 4],\ t \in [\pi, 2\pi]$; $r = 0.5,\ s \in [-4, 4],\ t \in [-\pi, \pi]$; $r = 0.5,\ s \in [0, 6\pi],\ t \in [0, 6\pi]$.

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