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ON AN INTEGRAL OPERATOR

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Abstract. In this paper we derive some criteria for univalence of an integral operator for analytic functions in the open unit disk.

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1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathcal{U}$.
Acu [1] had considered the integral operator $F_{\alpha,\beta,\gamma}$ given by

$$F_{\alpha,\beta,\gamma}(z) = \left[(1+\alpha+\beta+\gamma)\int_0^z (f(u))^\alpha (g(u))^\beta (h(u))^\gamma du\right]^{\frac{1}{1+\alpha+\beta+\gamma}}, \tag{1}$$

for α, β, γ be complex numbers, $\alpha+\beta+\gamma+1 \neq 0$, $f, g, h \in \mathcal{A}$ and obtain the next result, using Loewner chain.

Theorem 1. *Let $f, g, h \in \mathcal{A}$ and let α, β, γ be complex numbers with $|\alpha|+|\beta|+|\gamma| > 0$. If*

$$|\alpha+\beta+\gamma| < 1, \tag{2}$$

$$\left| |z|^2(\alpha+\beta+\gamma) + (1-|z|^2)\left(\alpha\frac{zf'(z)}{f(z)} + \beta\frac{zg'(z)}{g(z)} + \gamma\frac{zh'(z)}{h(z)}\right)\right| \leq 1, \tag{3}$$

for all $z \in \mathcal{U}$, then the function $F_{\alpha,\beta,\gamma}(z)$ given by (1) is analytic and univalent in $\mathcal{U}$.

In this paper we obtain new univalence criteria for the integral operator $F_{\alpha,\beta,\gamma}$ using Lemma Schwarz [3], Lemma Mocanu and Şerb [2] and Lemma Pascu [4].

2. Preliminary results

To discuss our problems for univalence of integral operator $F_{\alpha,\beta,\gamma}$ we need the following lemmas.

Lemma 2. *(Mocanu and Şerb [2]). Let* $M_0 = 1,5936...$ *the positive solution of equation*

$$(2-M)e^M = 2. \tag{4}$$

If $f \in \mathcal{A}$ *and*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0, \quad (z \in \mathcal{U}), \tag{5}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad (z \in \mathcal{U}). \tag{6}$$

The edge M_0 *is sharp.*

Lemma 3. *(Schwarz [3]). Let* $f(z)$ *be the function regular in the disk* $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ *with* $|f(z)| < M$, M *fixed. If* $f(z)$ *has in* $z = 0$ *one zero with multiplicity greater then* m, *then*

$$|f(z)| \le \frac{M}{R^m}|z|^m, \quad (z \in \mathcal{U}_R), \tag{7}$$

the equality (in the inequality (7) for $z \neq 0$*) can hold only if*

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ *is constant.*

Lemma 4. *(Pascu [4]). Let* δ *be a complex number,* $\mathrm{Re}\delta > 0$ *and the function* $f \in \mathcal{A}$*. If*

$$\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{8}$$

for all $z \in \mathcal{U}$*, then for every complex number* $\eta, \mathrm{Re}\eta \ge \mathrm{Re}\delta$*, the function*

$$F_\eta(z) = \left[\eta\int_0^z u^{\eta-1}f'(u)du\right]^{\frac{1}{\eta}} \tag{9}$$

is regular and univalent in $\mathcal{U}$*.*

3. Main results

Theorem 5. *Let $f, g, h \in \mathcal{A}$ and let δ be a complex number,* $\mathrm{Re}\delta > 0, K, L, M$ *real positive numbers.*
If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < K, \quad (z \in \mathcal{U}), \tag{10}$$

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < L, \quad (z \in \mathcal{U}), \tag{11}$$

$$\left|\frac{zh'(z)}{h(z)} - 1\right| < M, \quad (z \in \mathcal{U}), \tag{12}$$

and

$$|\alpha|K + |\beta|L + |\gamma|M \leq \frac{(2\mathrm{Re}\delta + 1)^{\frac{2\mathrm{Re}\delta+1}{2\mathrm{Re}\delta}}}{2}, \tag{13}$$

then for any complex numbers α, β, γ, $\mathrm{Re}(1+\alpha+\beta+\gamma) \geq \mathrm{Re}\delta$*, the integral operator $F_{\alpha,\beta,\gamma}$, defined by (1) is in the class $\mathcal{S}$.*

Proof. We have

$$F_{\alpha,\beta,\gamma}(z) = \left[(1+\alpha+\beta+\gamma)\int_0^z u^{\alpha+\beta+\gamma}\left(\frac{f(u)}{u}\right)^{\alpha}\left(\frac{g(u)}{u}\right)^{\beta}\left(\frac{h(u)}{u}\right)^{\gamma} du\right]^{\frac{1}{1+\alpha+\beta+\gamma}}. \tag{14}$$

Let's consider the function

$$p(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha}\left(\frac{g(u)}{u}\right)^{\beta}\left(\frac{h(u)}{u}\right)^{\gamma} du, \quad (z \in \mathcal{U}). \tag{15}$$

The function p is regular in $\mathcal{U}$ and $p(0) = p'(0) - 1 = 0$. We have

$$\frac{zp''(z)}{p'(z)} = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta\left(\frac{zg'(z)}{g(z)} - 1\right) + \gamma\left(\frac{zh'(z)}{h(z)} - 1\right), \ (z \in \mathcal{U}). \tag{16}$$

By (16) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left|\frac{zp''(z)}{p'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left[|\alpha|\left|\frac{zf'(z)}{f(z)} - 1\right| + |\beta|\left|\frac{zg'(z)}{g(z)} - 1\right| + |\gamma|\left|\frac{zh'(z)}{h(z)} - 1\right|\right], \tag{17}$$

for all $z \in \mathcal{U}$.
For (10), (11), (12) and Lemma 3 we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < K|z|, \tag{18}$$

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < L|z|, \tag{19}$$

$$\left|\frac{zh'(z)}{h(z)} - 1\right| < M|z|, \tag{20}$$

for all $z \in \mathcal{U}$, hence and (17) we obtain

$$\frac{1-|z|^{2\text{Re}\delta}}{\text{Re}\delta}\left|\frac{zp''(z)}{p'(z)}\right| \leq \frac{1-|z|^{2\text{Re}\delta}}{\text{Re}\delta}|z|[|\alpha|K + |\beta|L + |\gamma|M], \tag{21}$$

for all $z \in \mathcal{U}$.
Since

$$\max_{|z|\leq 1} \frac{1-|z|^{2\text{Re}\delta}}{\text{Re}\delta}|z| = \frac{2}{(2\text{Re}\delta + 1)^{\frac{2\text{Re}\delta+1}{2\text{Re}\delta}}},$$

by (13) and (21) we have

$$\frac{1-|z|^{2\text{Re}\delta}}{\text{Re}\delta}\left|\frac{zp''(z)}{p'(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{22}$$

From (15) we have $p'(z) = \left(\frac{f(z)}{z}\right)^{\alpha}\left(\frac{g(z)}{z}\right)^{\beta}\left(\frac{h(z)}{z}\right)^{\gamma}$, by (22) and Lemma 4 it results that $F_{\alpha,\beta,\gamma} \in \mathcal{S}$.

Theorem 6. *Let $f, g, h \in \mathcal{A}$ and let $\alpha, \beta, \gamma, \delta$ complex numbers,* $\text{Re}(1+\alpha+\beta+\gamma) \geq \text{Re}\delta > 0, K_i, L_i, M_i$ *positive real numbers,* $i = \overline{1,2}$.
If

$$|f(z)| < K_1; |g(z)| < L_1; |h(z)| < M_1, \tag{23}$$

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < K_2, \tag{24}$$

$$\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| < L_2, \tag{25}$$

$$\left|\frac{z^2 h'(z)}{h^2(z)} - 1\right| < M_2, \tag{26}$$

for all $z \in \mathcal{U}$ *and*

$$\begin{aligned}&[|\alpha|K_1K_2 + |\beta|L_1L_2 + |\gamma|M_1M_2]\mathrm{Re}\delta + \\ &+ [|\alpha|(K_1+1) + |\beta|(L_1+1) + |\gamma|(M_1+1)](\mathrm{Re}\delta+1)^{\frac{\mathrm{Re}\delta+1}{\mathrm{Re}\delta}} \leq \\ &\leq (\mathrm{Re}\delta+1)^{\frac{\mathrm{Re}\delta+1}{\mathrm{Re}\delta}} \cdot \mathrm{Re}\delta,\end{aligned} \tag{27}$$

then the integral operator $F_{\alpha,\beta,\gamma}$ *given by (1) belongs to the class* $\mathcal{S}$.

Proof. We consider the function

$$p(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} \left(\frac{g(u)}{u}\right)^{\beta} \left(\frac{h(u)}{u}\right)^{\gamma} du, \ (z \in \mathcal{U}). \tag{28}$$

From (16) we have

$$\begin{aligned}\frac{zp''(z)}{p'(z)} = \alpha\left[\left(\frac{z^2f'(z)}{f^2(z)} - 1\right)\frac{f(z)}{z} + \frac{f(z)}{z} - 1\right] + \beta\left[\left(\frac{z^2g'(z)}{g^2(z)} - 1\right)\frac{g(z)}{z} + \frac{g(z)}{z} - 1\right] + \\ + \gamma\left[\left(\frac{z^2h'(z)}{h^2(z)} - 1\right)\frac{h(z)}{z} + \frac{h(z)}{z} - 1\right], \ z \in \mathcal{U}\end{aligned}$$

and hence , we obtain

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left|\frac{zp''(z)}{p'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left[|\alpha|\left(\left|\frac{z^2f'(z)}{f^2(z)} - 1\right|\frac{|f(z)|}{|z|} + \frac{|f(z)|}{|z|} + 1\right)\right] + \\ + \frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left[|\beta|\left(\left|\frac{z^2g'(z)}{g^2(z)} - 1\right|\frac{|g(z)|}{|z|} + \frac{|g(z)|}{|z|} + 1\right)\right] + \\ + \frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left[|\gamma|\left(\left|\frac{z^2h'(z)}{h^2(z)} - 1\right|\frac{|h(z)|}{|z|} + \frac{|h(z)|}{|z|} + 1\right)\right]\end{aligned} \tag{29}$$

for all $z \in \mathcal{U}$.
From (23), (24), (25), (26) and Lemma 3 we get

$$|f(z)| \leq K_1|z|; |g(z)| \leq L_1|z|; |h(z)| \leq M_1|z|, \tag{30}$$

$$\left|\frac{z^2f'(z)}{f^2(z)} - 1\right| \leq K_2|z|^2, \tag{31}$$

$$\left|\frac{z^2g'(z)}{g^2(z)} - 1\right| \leq L_2|z|^2, \tag{32}$$

$$\left|\frac{z^2h'(z)}{h^2(z)} - 1\right| \leq M_2|z|^2, \tag{33}$$

for all $z \in \mathcal{U}$, and hence, by (29) we have

$$\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left|\frac{zp''(z)}{p'(z)}\right| \le \frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}|z|^2[|\alpha|K_1K_2+|\beta|L_1L_2+|\gamma|M_1M_2]+ \\ +\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}[|\alpha|(K_1+1)+|\beta|(L_1+1)+|\gamma|(M_1+1)], \tag{34}$$

for all $z \in \mathcal{U}$.
Since

$$\max_{z\le 1}\left[\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}|z|^2\right] = \frac{1}{(\mathrm{Re}\delta+1)^{\frac{\mathrm{Re}\delta+1}{\mathrm{Re}\delta}}},$$

from (34) and (27) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\left|\frac{zp''(z)}{p'(z)}\right| \le 1, \quad (z \in \mathcal{U}). \tag{35}$$

From (28) we have $p'(z) = \left(\frac{f(z)}{z}\right)^{\alpha}\left(\frac{g(z)}{z}\right)^{\beta}\left(\frac{h(z)}{z}\right)^{\gamma}$, by (35) and Lemma 4 we obtain that $F_{\alpha,\beta,\gamma} \in \mathcal{S}$.

Theorem 7. *Let $f, g, h \in \mathcal{A}$ and let α, β, γ be positive real numbers, M_0 the positive solution of equation $(2-M)e^M = 2, M_0 = 1,5936...$*
If

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0, \left|\frac{g''(z)}{g'(z)}\right| \le M_0, \left|\frac{h''(z)}{h'(z)}\right| \le M_0, \tag{36}$$

for all $z \in \mathcal{U}$, then the integral operator $F_{\alpha,\beta,\gamma} \in \mathcal{S}$.

Proof. Let us consider the function

$$p(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha}\left(\frac{g(u)}{u}\right)^{\beta}\left(\frac{h(u)}{u}\right)^{\gamma} du, \ (z \in \mathcal{U}). \tag{37}$$

From (16) we have

$$\frac{1-|z|^{2(\alpha+\beta+\gamma)}}{\alpha+\beta+\gamma}\left|\frac{zp''(z)}{p'(z)}\right| \le \frac{1-|z|^{2(\alpha+\beta+\gamma)}}{\alpha+\beta+\gamma}\left[\alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta\left|\frac{zg'(z)}{g(z)}-1\right|+\gamma\left|\frac{zh'(z)}{h(z)}-1\right|\right], \tag{38}$$

for all $z \in \mathcal{U}$.
By (36) and Lemma 2 we obtain

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1, \ \left|\frac{zg'(z)}{g(z)}-1\right| < 1, \ \left|\frac{zh'(z)}{h(z)}-1\right| < 1,$$

for all $z \in \mathcal{U}$ and hence, by (38) we get

$$\frac{1-|z|^{2(\alpha+\beta+\gamma)}}{\alpha+\beta+\gamma}\left|\frac{zp''(z)}{p'(z)}\right| \leq 1, \ (z \in \mathcal{U}). \tag{39}$$

From (37) we have $p'(z) = \left(\frac{f(z)}{z}\right)^{\alpha}\left(\frac{g(z)}{z}\right)^{\beta}\left(\frac{h(z)}{z}\right)^{\gamma}$, by Lemma 4,
for $1+\alpha+\beta+\gamma = \mathrm{Re}\delta = \mathrm{Re}\eta$, it results that $F_{\alpha,\beta,\gamma} \in \mathcal{S}$.

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