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REMARKS ON EXTENDED GAUSS HYPERGEOMETRIC FUNCTIONS

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Abstract. Recently, Singh [1] established some interesting results for the extended Gauss hypergeometric function (EGHF) due to Özergin et al. [7]. Motivated by the work of Singh [1], in this paper, we further establish some interesting theorems for the extended Gauss hypergeometric function (EGHF) defined by Srivastava et al. [9]. Some deductions of our main results are also considered.

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1. Introduction

In recent years, a number of authors namely, Chaudhry et al. [11], Chaudhry et al. [12], Lee et al. [2], Özergin et al. [7], Parmar [14], Srivastava et al. [9], Liu and Wang [8], Khan and Ghayasuddin [13], Choi et al. [10] etcetera have introduced and investigated various extension of some well-known special functions.

In particular, among several interesting and potentially useful properties of the extended Gauss hypergeometric functions, very recently, Singh [1] derived some other interesting results for the extended Gauss hypergeometric function (EGHF) defined by Özergin et al. [7]. In a sequel of above-mentioned works, in the present note, we further derive various (presumably) new and potentially useful properties of the extended Gauss hypergeometric function (EGHF) defined by Srivastava et al. [9].

For the purposes of our present study, we begin by recalling here the following definitions of some known special functions:

Recently, Srivastava et al. [9] introduced a new generalization of Gauss hypergeometric function as follows:

$$F_p^{(\alpha,\beta;k,\mu)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;k,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \tag{1}$$

$$(|z| < 1;\ \min\{\Re(\alpha),\ \Re(\beta),\ \Re(\mathrm{k}),\ \Re(\mu)\} > 0;\ \Re(c) > \Re(b) > 0;\ \Re(p) \geq 0),$$

where $B_p^{(\alpha,\beta;k,\mu)}$ is the generalized beta function defined as follows:

$$B_p^{(\alpha,\beta;k,\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^k(1-t)^\mu}\right)dt \tag{2}$$

$$(\Re(p) \geq 0;\ \min\{\Re(\mathrm{x}),\ \Re(\mathrm{y}),\ \Re(\alpha),\ \Re(\beta)\} > 0;\ \min\{\Re(k),\ \Re(\mu)\} > 0).$$

By using (2), we can obtain the following integral representation of the extended Gauss hypergeometric function $F_p^{(\alpha,\beta;k,\mu)}$:

$$F_p^{(\alpha,\beta;k,\mu)}(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^k(1-t)^\mu}\right)dt \tag{3}$$

$$(\Re(p) \geq 0;\ |\arg(1-z)| < \pi;\ \Re(c) > \Re(b) > 0;\ \min\{\Re(k),\ \Re(\mu)\} > 0).$$

On setting $k = \mu$ in (3), we get the extended Gauss hypergeometric function defined by Parmar [14], which further gives the known generalization of Gauss hypergeometric function given by Özergin et al. [7] by taking $\mu = 1$. Also, it is noticed that, if we set $\alpha = \beta$ and $k = \mu$ in (3) then we get the extended Gauss hypergeometric function defined by Lee et al. [2] and if we set $\alpha = \beta$ and $k = \mu = 1$ in (3) then we get the extended Gauss hypergeometric function defined by Chaudhry et al. [12].

For $p = 0$, (3) reduces obviously to the classical Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ (see [3]).

The generalized Wright hypergeometric function is defined by (see [4], [5] and [6])

$${}_p\Psi_q\left[\begin{array}{cccc}(\alpha_1\ ,\ A_1), & \ldots.., & (\alpha_p\ ,\ A_p); & \\ & & & z\\ (\beta_1\ ,\ B_1), & \ldots.., & (\beta_q\ ,\ B_q); & \end{array}\right] = \sum_{k=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j + A_j k)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j + B_j k)}\frac{z^k}{k!}, \tag{4}$$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{5}$$

2. Main Results

This section deals with some interesting results for the various extended Gauss hypergeometric functions.

Theorem 1. *The following result holds true: For* $\Re(\alpha) > 0$*,* $\Re(\beta) > 0$ *and* $\Re(c) > \Re(b) > 0$*,*

$$F_p^{(\alpha,\beta;k,\mu)}(a,b;c;1) = \frac{\Gamma(\beta)\Gamma(c)}{\Gamma(\alpha)\Gamma(b)\Gamma(c-b)} \times {}_3\Psi_2\left[\begin{array}{c} (\alpha,1), \quad (b,-k), \quad (c-b-a,-\mu); \\ \\ (\beta,1), \quad (c-a,-k-\mu); \end{array} -p\right], \tag{6}$$

where $F_p^{(\alpha,\beta;k,\mu)}$ *is the EGHF defined by Srivastava et al. [9] and* ${}_3\Psi_2$ *is the Wright hypergeometric function defined by (4) satisfied the condition (5).*

Proof. Using (3) on the left-hand side of (6), to get

$$F_p^{(\alpha,\beta;k,\mu)}(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^k(1-t)^\mu}\right)dt. \tag{7}$$

On expanding ${}_1F_1$ in its defining series, changing the order of summation and integration (which is guaranteed under the conditions), we obtain

$$F_p^{(\alpha,\beta;k,\mu)}(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\beta)_r}\frac{(-p)^r}{r!} \times \int_0^1 t^{b-kr-1}(1-t)^{c-b-a-\mu r-1}dt. \tag{8}$$

Evaluating the above integral with the help of beta function and after some simplification, we arrive at

$$F_p^{(\alpha,\beta;k,\mu)}(a,b;c;1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)} \times \sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\Gamma(b-kr)\Gamma(c-b-a-\mu r)}{\Gamma(\beta+r)\Gamma(c-a-kr-\mu r)}\frac{(-p)^r}{r!}, \tag{9}$$

which upon using the definition (4), yields (6). This completes the proof.

Remark 1. *If we set $k = \mu$ in (6) then we obtain the following result for the EGHF defined by Parmar [14]:*

$$F_p^{(\alpha,\beta;\mu,\mu)}(a,b;c;1) = F_p^{(\alpha,\beta;\mu)}(a,b;c;1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)}$$

$$\times\, {}_3\Psi_2\left[\begin{array}{c} (\alpha,1),\quad (b,-\mu),\quad (c-b-a,-\mu); \\ \\ (\beta,1),\quad (c-a,-2\mu); \end{array} -p\right]. \tag{10}$$

Remark 2. *If we set $k = \mu = 1$ in (6) then we get the following known result of Singh [1, p.2, Theorem 2.1] for the EGHF defined by Özergin et al. [7]:*

$$F_p^{(\alpha,\beta;1,1)}(a,b;c;1) = F_p^{(\alpha,\beta)}(a,b;c;1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)}$$

$$\times\, {}_3\Psi_2\left[\begin{array}{c} (\alpha,1),\quad (b,-1),\quad (c-b-a,-1); \\ \\ (\beta,1),\quad (c-a,-2); \end{array} -p\right]. \tag{11}$$

Remark 3. *If we consider $\beta = \alpha$ and $k = \mu$ in (6) then we obtain the following result for the EGHF defined by Lee et al. [2]:*

$$F_p^{(\alpha,\alpha;\mu,\mu)}(a,b;c;1) = F_p^{\mu}(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\times\, {}_2\Psi_1\left[\begin{array}{c} (b,-\mu),\quad (c-b-a,-\mu); \\ \\ (c-a,-2\mu); \end{array} -p\right]. \tag{12}$$

Remark 4. *If we put $\beta = \alpha$ and $k = \mu = 1$ in (6) then we obtain the following known result of Singh [1, p.3, Eq.(2.5)] for the EGHF defined by Chaudhry et al. [12]:*

$$F_p^{(\alpha,\alpha;1,1)}(a,b;c;1) = F_p(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\times\, {}_2\Psi_1\left[\begin{array}{c} (b,-1),\quad (c-b-a,-1); \\ \\ (c-a,-2); \end{array} -p\right]. \tag{13}$$

Remark 5. *On setting $p = 0$ in (9), after little simplification, we get the following known result due to Rainville [3, p.49]:*

$$F_0^{(\alpha,\beta;k,\mu)}(a,b;c;1) = {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}. \tag{14}$$

Theorem 2. *The following result holds true: For $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\Re(c) > \Re(a+n) > 0$ (n is non-negative integer),*

$$F_p^{(\alpha,\beta;k,\mu)}(-n, a+n; c; 1) = \frac{\Gamma(\beta)\Gamma(c)}{\Gamma(\alpha)\Gamma(a+n)\Gamma(c-a-n)} \times {}_3\Psi_2 \left[\begin{array}{ccc} (\alpha,1), & (a+n,-k), & (c-a,-\mu); \\ & (\beta,1), & (c+n,-k-\mu); \end{array} -p \right], \tag{15}$$

where $F_p^{(\alpha,\beta;k,\mu)}$ is the EGHF defined by Srivastava et al. [9] and ${}_3\Psi_2$ is the Wright hypergeometric function defined by (4) satisfied the condition (5).

Proof. On using (3) on the left-hand side of (15), expanding ${}_1F_1$ in its defining series, and changing the order of summation and integration (which is guaranteed under the conditions), and after little simplification, we get

$$F_p^{(\alpha,\beta;k,\mu)}(-n, a+n; c; 1) = \frac{\Gamma(c)}{\Gamma(a+n)\Gamma(c-a-n)} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r} \frac{(-p)^r}{r!} \times \int_0^1 t^{a+n-kr-1}(1-t)^{c-a-\mu r-1} dt. \tag{16}$$

Evaluating the above integral with the help of beta function and after some simplification, we get

$$F_p^{(\alpha,\beta;k,\mu)}(-n, a+n; c; 1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(a+n)\Gamma(c-a-n)\Gamma(\alpha)} \times \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\Gamma(a+n-kr)\Gamma(c-a-\mu r)}{\Gamma(\beta+r)\Gamma(c+n-kr-\mu r)} \frac{(-p)^r}{r!}, \tag{17}$$

which upon using the definition (4), yields (15). This completes the proof.

Remark 6. *If we set $k = \mu$ in (15) then we obtain the following result for the EGHF defined by Parmar [14]:*

$$F_p^{(\alpha,\beta;\mu,\mu)}(-n, a+n; c; 1) = F_p^{(\alpha,\beta;\mu)}(-n, a+n; c; 1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(a+n)\Gamma(c-a-n)\Gamma(\alpha)} \times {}_3\Psi_2 \left[\begin{array}{ccc} (\alpha,1), & (a+n,-\mu), & (c-a,-\mu); \\ & (\beta,1), & (c+n,-2\mu); \end{array} -p \right]. \tag{18}$$

Remark 7. *If we set $k = \mu = 1$ in (15) then we get the following known result of Singh [1, p.3, Theorem 2.5] for the EGHF defined by Özergin et al. [7]:*

$$F_p^{(\alpha,\beta;1,1)}(-n, a+n; c; 1) = F_p^{(\alpha,\beta)}(-n, a+n; c; 1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(a+n)\Gamma(c-a-n)\Gamma(\alpha)}$$

$$\times\, {}_3\Psi_2 \left[\begin{array}{ccc} (\alpha,1), & (a+n,-1), & (c-a,-1); \\ & (\beta,1), & (c+n,-2); \end{array} -p \right]. \tag{19}$$

Remark 8. *If we consider $\beta = \alpha$ and $k = \mu$ in (15) then we obtain the following result for the EGHF defined by Lee et al. [2]:*

$$F_p^{(\alpha,\alpha;\mu,\mu)}(-n, a+n; c; 1) = F_p^{\mu}(-n, a+n; c; 1) = \frac{\Gamma(c)}{\Gamma(a+n)\Gamma(c-a-n)}$$

$$\times\, {}_2\Psi_1 \left[\begin{array}{cc} (a+n,-\mu), & (c-a,-\mu); \\ & (c+n,-2\mu); \end{array} -p \right]. \tag{20}$$

Remark 9. *If we consider $\beta = \alpha$ and $k = \mu = 1$ in (15) then we get the following known result of Singh [1, p.4, Eq.(2.10)] for the EGHF defined by Chaudhry et al. [12]:*

$$F_p^{(\alpha,\alpha;1,1)}(-n, a+n; c; 1) = F_p(-n, a+n; c; 1) = \frac{\Gamma(c)}{\Gamma(a+n)\Gamma(c-a-n)}$$

$$\times\, {}_2\Psi_1 \left[\begin{array}{cc} (a+n,-1), & (c-a,-1); \\ & (c+n,-2); \end{array} -p \right]. \tag{21}$$

Remark 10. *On setting $p = 0$ in (17), after little simplification, we get the following known result due to Rainville [3, p.69]:*

$$F_0^{(\alpha,\beta;k,\mu)}(-n, a+n; c; 1) = {}_2F_1(-n, a+n; c; 1) = \frac{(-1)^n(1-c+a)_n}{(c)_n}. \tag{22}$$

Theorem 3. *The following result holds true: For $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(b) > 0$ and $\Re(\frac{1}{2} - \frac{n}{2}) > 0$ (n is non-negative integer),*

$$F_p^{(\alpha,\beta;k,\mu)}\left[-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; b+\frac{1}{2}; 1\right] = \frac{2^n\Gamma(2b)\Gamma(b+\frac{n}{2}+\frac{1}{2})\Gamma(\beta)}{\Gamma(b)\Gamma(2b+n)\Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(\alpha)}$$

$$\times\, {}_3\Psi_2\left[\begin{array}{lll}(\alpha,1), & (\frac{1}{2}-\frac{n}{2},-k), & (b+n,-\mu); \\ & (\beta,1), & (b+\frac{n}{2}+\frac{1}{2},-k-\mu);\end{array} -p\right], \tag{23}$$

where $F_p^{(\alpha,\beta;k,\mu)}$ *is the EGHF defined by Srivastava et al. [9] and* ${}_3\Psi_2$ *is the Wright hypergeometric function defined by (4) satisfied the condition (5).*

Proof. On applying (3) on the left-hand side of (23), expanding ${}_1F_1$ in its defining series, and changing the order of summation and integration (which is guaranteed under the conditions), and after little simplification, we get

$$F_p^{(\alpha,\beta;k,\mu)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right]=\frac{\Gamma(b+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(b+\frac{n}{2})}$$

$$\times\sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\beta)_r}\frac{(-p)^r}{r!}\int_0^1 t^{\frac{1}{2}-\frac{n}{2}-kr-1}(1-t)^{b+n-\mu r-1}dt. \tag{24}$$

Evaluating the above integral with the help of beta function and after some simplification, we get

$$F_p^{(\alpha,\beta;k,\mu)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right]=\frac{\Gamma(b+\frac{1}{2})\Gamma(\beta)}{\Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(b+\frac{n}{2})\Gamma(\alpha)}$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\Gamma(\frac{1}{2}-\frac{n}{2}-kr)\Gamma(b+n-\mu r)}{\Gamma(\beta+r)\Gamma(b+\frac{n}{2}+\frac{1}{2}-kr-\mu r)}\frac{(-p)^r}{r!}. \tag{25}$$

By using the Legendre's duplication formula, $\Gamma(b)\Gamma(b+\frac{1}{2})=2^{1-2b}\pi\Gamma(2b)$, in the above equation, we get

$$F_p^{(\alpha,\beta;k,\mu)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right]=\frac{2^n\Gamma(2b)\Gamma(b+\frac{n}{2}+\frac{1}{2})\Gamma(\beta)}{\Gamma(b)\Gamma(2b+n)\Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(\alpha)}$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\Gamma(\frac{1}{2}-\frac{n}{2}-kr)\Gamma(b+n-\mu r)}{\Gamma(\beta+r)\Gamma(b+\frac{n}{2}+\frac{1}{2}-kr-\mu r)}\frac{(-p)^r}{r!}. \tag{26}$$

which upon using the definition (4), yields (23). This completes the proof.

Remark 11. *If we set* $k=\mu$ *in (23) then we obtain the following result for the EGHF defined by Parmar [14]:*

$$F_p^{(\alpha,\beta;\mu,\mu)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right]=F_p^{(\alpha,\beta;\mu)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right]$$

$$= \frac{2^n\Gamma(2b)\Gamma(b+\frac{n}{2}+\frac{1}{2})\Gamma(\beta)}{\Gamma(b)\Gamma(2b+n)\Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(\alpha)}\ {}_3\Psi_2\left[\begin{array}{c}(\alpha,1),\ (\frac{1}{2}-\frac{n}{2},-\mu),\ (b+n,-\mu);\\ (\beta,1),\ (b+\frac{n}{2}+\frac{1}{2},-2\mu);\end{array} -p\right]. \tag{27}$$

Remark 12. *If we put $k=\mu=1$ in (23) then we get the following known result of Singh [1, p.4, Theorem 2.9] for the EGHF defined by Özergin et al. [7]:*

$$F_p^{(\alpha,\beta;1,1)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right] = F_p^{(\alpha,\beta)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right]$$

$$= \frac{2^n\Gamma(2b)\Gamma(b+\frac{n}{2}+\frac{1}{2})\Gamma(\beta)}{\Gamma(b)\Gamma(2b+n)\Gamma(\frac{1}{2}-\frac{n}{2})\Gamma(\alpha)}\ {}_3\Psi_2\left[\begin{array}{c}(\alpha,1),\ (\frac{1}{2}-\frac{n}{2},-1),\ (b+n,-1);\\ (\beta,1),\ (b+\frac{n}{2}+\frac{1}{2},-2);\end{array} -p\right]. \tag{28}$$

Remark 13. *If we set $\beta=\alpha$ and $k=\mu$ in (23) then we get the following result for the EGHF defined by Lee et al. [2]:*

$$F_p^{(\alpha,\alpha;\mu,\mu)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right] = F_p^{\mu}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right]$$

$$= \frac{2^n\Gamma(2b)\Gamma(b+\frac{n}{2}+\frac{1}{2})}{\Gamma(b)\Gamma(2b+n)\Gamma(\frac{1}{2}-\frac{n}{2})}\ {}_2\Psi_1\left[\begin{array}{c}(\frac{1}{2}-\frac{n}{2},-\mu),\ (b+n,-\mu);\\ (b+\frac{n}{2}+\frac{1}{2},-2\mu);\end{array} -p\right]. \tag{29}$$

Remark 14. *If we set $\beta=\alpha$ and $k=\mu=1$ in (23) then we get the following known result of Singh [1, p.6, Eq.(2.16)] for the EGHF defined by Chaudhry et al. [12]:*

$$F_p^{(\alpha,\alpha;1,1)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right] = F_p\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right]$$

$$= \frac{2^n\Gamma(2b)\Gamma(b+\frac{n}{2}+\frac{1}{2})}{\Gamma(b)\Gamma(2b+n)\Gamma(\frac{1}{2}-\frac{n}{2})}\ {}_2\Psi_1\left[\begin{array}{c}(\frac{1}{2}-\frac{n}{2},-1),\ (b+n,-1);\\ (b+\frac{n}{2}+\frac{1}{2},-2);\end{array} -p\right]. \tag{30}$$

Remark 15. *On setting $p=0$ in (26), after little simplification, we get the following known result due to Rainville [3, p.50]:*

$$F_0^{(\alpha,\beta;k,\mu)}\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right] = {}_2F_1\left[-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};b+\frac{1}{2};1\right] = \frac{2^n(b)_n}{(2b)_n}. \tag{31}$$

Theorem 4. *The following result holds true: For* $\Re(\alpha) > 0$, $\Re(\beta) > 0$ *and* $\Re(c) > \Re(1-b-n) > 0$ *(n is non-negative integer),*

$$F_p^{(\alpha,\beta;k,\mu)}(-n, 1-b-n; c; 1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(1-b-n)\Gamma(c-1+b+n)\Gamma(\alpha)}$$

$$\times {}_3\Psi_2 \left[\begin{array}{ccc} (\alpha,1), & (1-b-n,-k), & (c-1+b+2n,-\mu); \\ & & \\ (\beta,1), & & (c+n,-k-\mu); \end{array} -p \right], \tag{32}$$

where $F_p^{(\alpha,\beta;k,\mu)}$ *is the EGHF defined by Srivastava et al. [9] and* ${}_3\Psi_2$ *is the Wright hypergeometric function defined by (4) satisfied the condition (5).*

Proof. On using (3) on the left-hand side of (32), expanding ${}_1F_1$ in its defining series, changing the order of summation and integration (which is guaranteed under the conditions), and after little simplification, we get

$$F_p^{(\alpha,\beta;k,\mu)}(-n, 1-b-n; c; 1) = \frac{\Gamma(c)}{\Gamma(1-b-n)\Gamma(c-1+b+n)}$$

$$\times \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r} \frac{(-p)^r}{r!} \int_0^1 t^{1-b-n-kr-1}(1-t)^{c-1+b+2n-\mu r-1} dt. \tag{33}$$

Evaluating the above integral with the help of beta function and after some simplification, we get

$$F_p^{(\alpha,\beta;k,\mu)}(-n, 1-b-n; c; 1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(1-b-n)\Gamma(c-1+b+n)\Gamma(\alpha)}$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(1-b-n-kr)\Gamma(c-1+b+2n-\mu r)\Gamma(\alpha+r)}{\Gamma(c+n-\mu r-kr)\Gamma(\beta+r)} \frac{(-p)^r}{r!}, \tag{34}$$

which upon using the definition (4), yields (32). This completes the proof.

Remark 16. *If we set* $k = \mu$ *in (32) then we get the following result for the EGHF defined by Parmar [14]:*

$$F_p^{(\alpha,\beta;\mu,\mu)}(-n, 1-b-n; c; 1) = F_p^{(\alpha,\beta;\mu)}(-n, 1-b-n; c; 1)$$

$$= \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(1-b-n)\Gamma(c-1+b+n)\Gamma(\alpha)}$$

$$\times {}_3\Psi_2 \left[\begin{array}{ccc} (\alpha,1), & (1-b-n,-\mu), & (c-1+b+2n,-\mu); \\ & & \\ (\beta,1), & & (c+n,-2\mu); \end{array} -p \right]. \tag{35}$$

Remark 17. *If we set $k=\mu=1$ in (32) then we get the following known result of Singh [1, p.6, Theorem 2.13] for the EGHF defined by Özergin et al. [7]:*

$$F_p^{(\alpha,\beta;1,1)}(-n,1-b-n;c;1)=F_p^{(\alpha,\beta)}(-n,1-b-n;c;1)$$

$$=\frac{\Gamma(c)\Gamma(\beta)}{\Gamma(1-b-n)\Gamma(c-1+b+n)\Gamma(\alpha)}$$

$$\times\,{}_3\Psi_2\left[\begin{array}{ccc}(\alpha,1), & (1-b-n,-1), & (c-1+b+2n,-1);\\ & & & -p\\ & (\beta,1), & (c+n,-2);\end{array}\right]. \quad (36)$$

Remark 18. *If we set $\beta=\alpha$ and $k=\mu$ in (32) then we get the following result for the EGHF defined by Lee et al. [2]:*

$$F_p^{(\alpha,\alpha;\mu,\mu)}(-n,1-b-n;c;1)=F_p^{\mu}(-n,1-b-n;c;1)=\frac{\Gamma(c)}{\Gamma(1-b-n)\Gamma(c-1+b+n)}$$

$$\times\,{}_2\Psi_1\left[\begin{array}{cc}(1-b-n,-\mu), & (c-1+b+2n,-\mu);\\ & & -p\\ & (c+n,-2\mu);\end{array}\right]. \quad (37)$$

Remark 19. *If we consider $\beta=\alpha$ and $k=\mu=1$ in (32) then we get the following known result of Singh [1, p.7, Eq.(2.21)] for the EGHF defined by Chaudhry et al. [12]:*

$$F_p^{(\alpha,\alpha;1,1)}(-n,1-b-n;c;1)=F_p(-n,1-b-n;c;1)=\frac{\Gamma(c)}{\Gamma(1-b-n)\Gamma(c-1+b+n)}$$

$$\times\,{}_2\Psi_1\left[\begin{array}{cc}(1-b-n,-1), & (c-1+b+2n,-1);\\ & & -p\\ & (c+n,-2);\end{array}\right]. \quad (38)$$

Remark 20. *If we set $p=0$ and $c=a$ in (34), after little simplification, we get the following known result due to Rainville [3, p.69]:*

$$F_0^{(\alpha,\beta;k,\mu)}(-n,1-b-n;a;1)={}_2F_1(-n,1-b-n;a;1)=\frac{(a+b-1)_{2n}}{(a)_n(a+b-1)_n}. \quad (39)$$

Theorem 5. *The following result holds true: For $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(c)>\Re(b)>0$,*

$$F_p^{(\alpha,\beta;k,\mu)}(-n,b;c;1)=\frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)}$$

$$\times\, {}_3\Psi_2\left[\begin{array}{ccc}(\alpha,1), & (b,-k), & (c-b+n,-\mu); \\ & (\beta,1), & (c+n,-k-\mu);\end{array} -p\right], \tag{40}$$

where n is non-negative integer, $F_p^{(\alpha,\beta;k,\mu)}$ is the EGHF defined by Srivastava et al. [9] and ${}_3\Psi_2$ is the Wright hypergeometric function defined by (4) satisfied the condition (5).

Proof. On using (3) on the left-hand side of (40), expanding ${}_1F_1$ in its defining series, changing the order of summation and integration (which is guaranteed under the conditions), and after little simplification, we get

$$F_p^{(\alpha,\beta;k,\mu)}(-n,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\times \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r}\frac{(-p)^r}{r!}\int_0^1 t^{b-kr-1}(1-t)^{c-b+n-\mu r-1}dt. \tag{41}$$

In the above equation, using the definition of beta function and after some simplification, equation (41) reduces to

$$F_p^{(\alpha,\beta;k,\mu)}(-n,b;c;1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)}$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(b-kr)\Gamma(c-b+n-\mu r)\Gamma(\alpha+r)}{\Gamma(c+n-\mu r-kr)\Gamma(\beta+r)}\frac{(-p)^r}{r!}, \tag{42}$$

which upon using the definition (4), yields (40). This completes the proof.

Remark 21. *If we set $k=\mu$ in (40) then we get the following result for the EGHF defined by Parmar [14]:*

$$F_p^{(\alpha,\beta;\mu,\mu)}(-n,b;c;1) = F_p^{(\alpha,\beta;\mu)}(-n,b;c;1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)}$$

$$\times\, {}_3\Psi_2\left[\begin{array}{ccc}(\alpha,1), & (b,-\mu), & (c-b+n,-\mu); \\ & (\beta,1), & (c+n,-2\mu);\end{array} -p\right]. \tag{43}$$

Remark 22. *If we set $k=\mu=1$ in (40) then we get the following known result of Singh [1, p.7, Theorem 2.17] for the EGHF defined by Özergin et al. [7]:*

$$F_p^{(\alpha,\beta;1,1)}(-n,b;c;1) = F_p^{(\alpha,\beta)}(-n,b;c;1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)}$$

$$\times\,{}_3\Psi_2\left[\begin{array}{c}(\alpha,1),\quad (b,-1),\quad (c-b+n,-1);\\ \\ (\beta,1),\qquad (c+n,-2);\end{array}\;-p\right].\tag{44}$$

Remark 23. *If we put $\beta=\alpha$ and $k=\mu$ in (40) then we get the following result for the EGHF defined by Lee et al. [2]:*

$$F_p^{(\alpha,\alpha;\mu,\mu)}(-n,b;c;1)=F_p^{\mu}(-n,b;c;1)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\times\,{}_2\Psi_1\left[\begin{array}{c}(b,-\mu),\quad (c-b+n,-\mu);\\ \\ (c+n,-2\mu);\end{array}\;-p\right].\tag{45}$$

Remark 24. *If we set $\beta=\alpha$ and $k=\mu=1$ in (40) then we obtain the following known result of Singh [1, p.8, Eq.(2.26)] for the EGHF defined by Chaudhry et al. [12]:*

$$F_p^{(\alpha,\alpha;1,1)}(-n,b;c;1)=F_p(-n,b;c;1)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}$$

$$\times\,{}_2\Psi_1\left[\begin{array}{c}(b,-1),\quad (c-b+n,-1);\\ \\ (c+n,-2);\end{array}\;-p\right].\tag{46}$$

Remark 25. *If we consider $p=0$ in (42), after some simplification, we get the following known result due to Rainville [3, p.69]:*

$$F_0^{(\alpha,\beta;k,\mu)}(-n,b;c;1)={}_2F_1(-n,b;c;1)=\frac{(c-b)_n}{(c)_n}.\tag{47}$$

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