CERTAIN PROPERTIES OF AN INTEGRAL OPERATOR

V. Pescar

Abstract. In this paper we consider the integral operator Miller-Mocanu-Reade for analytic functions in the open unit disk and we obtain properties of this integral operator.

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1. Introduction

Let $A$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $S$ the subclass of $A$ consisting of functions $f \in A$, which are univalent in $U$.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in $U$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} - \{0\}$ we note

$$H[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + \cdots\}$$

and

$$A_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots\},$$

with $A_1 = A$.

Let us denote $S_\alpha(\rho)$ the class spiral functions of type $\alpha$ and order $\rho$, where $\alpha, \rho \in \mathbb{R}$,

$$S_\alpha(\rho) = \left\{ f \in A : \Re \frac{e^{i\alpha}zf'(z)}{f(z)} > \rho \cos \alpha, |\alpha| < \frac{\pi}{2}, \rho < 1, z \in U \right\}.$$
We have $S_\alpha(0) = S_\alpha$, where $S_\alpha$ is the class spiral functions of type $\alpha$.

In this paper we consider the integral operator Miller-Mocanu-Reade, $I_{\alpha, \beta, \gamma, \delta} : E \rightarrow \mathcal{H}(U), E \subseteq \mathcal{H}(U)$ defined by:

$$I_{\alpha, \beta, \gamma, \delta}(f)(z) = \left[ \frac{\beta + \gamma}{z^\gamma \varphi(z)} \int_0^z f^\alpha(t)t^{\delta-1}\varphi(t)dt \right]^{1/\beta},$$  \hspace{1cm} (5)

where $\phi, \varphi \in H[1, n]$ with $\phi(z)\varphi(z) \neq 0, z \in U, \alpha, \beta, \gamma, \delta \in \mathbb{C}, \beta \neq 0, \alpha + \delta = \beta + \gamma$ and $\Re(\alpha + \delta) > 0, f \in A_n, f(z) = z + \alpha_{n+1}z^{n+1} + \cdots, n \in \mathbb{N} - \{0\}$.

The integral operator $I_{\alpha, \gamma, \delta}$ was defined by S.S. Miller, P.T. Mocanu and M.O. Reade in 1978 [1] and studied in [2], [3], [4], [5].

For $\alpha = \beta = e^{i\sigma}, \sigma \in \mathbb{R}, \gamma = \delta, f \in S_\sigma(\rho), \phi(z) = \varphi(z) = 1, z \in U$, from (5) we obtain the integral operator

$$T_{\gamma, \sigma}(z) = \left[ \frac{e^{i\sigma} + \gamma}{z^\gamma} \int_0^z [f(t)]^{e^{i\sigma}} t^{\gamma-1}dt \right]^{e^{-i\sigma}},$$  \hspace{1cm} (6)

for all $z \in U$, that was studied by S.K. Bajpai in 1979 [7], which proved that if $f \in S_\sigma(\rho), 0 \leq \rho < 1, \Re \gamma > -\rho \cos \sigma, |\sigma| < \frac{\pi}{2}$, then $T_{\gamma, \sigma} \in S_\sigma(\rho)$.

From (5), for $\alpha = \beta, \gamma = 0, \delta = 0, f \in A, \phi(z) = \varphi(z) = 1, z \in U$, we obtain the integral operator Miller-Mocanu [8],

$$J_\beta(z) = \left[ \beta \int_0^z t^{-1}f^\beta(t)dt \right]^{1/\beta}, \hspace{1cm} z \in U.$$  \hspace{1cm} (7)

In this paper we obtain properties of integral operator $I_{\alpha, \beta, \gamma, \delta}(f)$.

2. Preliminaries

We need the following lemmas.

**Lemma 1.** Pascu [9]. Let $\alpha$ be a complex number, $\Re \alpha > 0$ and $f \in A$. If

$$\frac{1 - |z|^{2\Re \alpha}}{\Re \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$  \hspace{1cm} (8)

for all $z \in U$, then the function

$$F_\alpha(z) = \left[ \alpha \int_0^z t^{\alpha-1}f'(t)dt \right]^{1/\alpha}$$  \hspace{1cm} (9)

is regular and univalent in $U$. 
Lemma 2. General Schwarz Lemma [10]. Let \( f \) the function regular in the disk \( U_R = \{ z \in \mathbb{C} : |z| < R \} \), with \( |f(z)| < M \), \( M \) fixed. If the function \( f \) has in \( z = 0 \) one zero with multiply \( \geq m \), then

\[
|f(z)| \leq \frac{M}{R^m}|z|^m, \quad z \in U_R, \quad (10)
\]

the equality (in the inequality (10) for \( z \neq 0 \)) can hold only if

\[
f(z) = e^{i\theta} \frac{M}{R^m}z^m,
\]

where \( \theta \) is constant.

3. Main results

Theorem 3. Let \( \alpha, \beta, \gamma, \delta \) be complex numbers, \( \beta \neq 0, \beta + \gamma = \alpha + \delta \neq 0, \)

\[
a = \text{Re}(\alpha + \delta) > 0
\]

and the functions \( \phi, \varphi \in H[1, n] \) with \( \phi(z)\varphi(z) \neq 0, z \in U \), the function \( f \in A_n, f(z) = z + a_n z^{n+1} + \cdots, L, M \) positive real numbers.

If

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < M, \quad z \in U, \quad (11)
\]

\[
\left| \frac{z\varphi'(z)}{\varphi(z)} \right| < L, \quad z \in U \quad (12)
\]

and

\[
|a|M + L \leq \frac{(2a + n)^{\frac{2a+n}{2a}}}{2n^{\frac{n}{2a}}}, \quad n \in \mathbb{N} - \{0\}, \quad (13)
\]

then

\[
I_{\alpha,\beta,\gamma,\delta}(f)(z) = \frac{1}{\varphi^{\beta}(z)} z \left( 1 + b_2 z + b_3 z^2 + \cdots \right)^{\frac{\gamma}{\beta}}, \quad z \in U \quad (14)
\]

and

\[
\frac{1}{z^{\frac{\beta}{\beta+\gamma}} \phi^{\frac{\beta}{\beta+\gamma}}(z)} I_{\alpha,\beta,\gamma,\delta}^{\beta}(f)(z) \quad (15)
\]

belongs to class \( S \).
Proof. From (5) we have

\[ I_{\alpha,\beta,\gamma,\delta}(f)(z) = \left[ \frac{\beta + \gamma}{(\alpha + \delta)z^\gamma \phi(z)} \right]^{\frac{1}{\beta}} \left\{ (\alpha + \delta) \int_0^z t^{\alpha + \delta - 1} \left( \frac{f(t)}{t} \right)^\alpha \varphi(t) \, dt \right\}^{\frac{1}{\alpha + \delta}} \]  

(16)

for all \( z \in U \).

We consider the function

\[ G_{\alpha,\delta}(z) = \left[ (\alpha + \delta) \int_0^z t^{\alpha + \delta - 1} \left( \frac{f(t)}{t} \right)^\alpha \varphi(t) \, dt \right]^{\frac{1}{\alpha + \delta}}, \quad z \in U, \]  

(17)

where \( \alpha + \delta = \beta + \gamma \).

Let the function

\[ p(z) = \int_0^z \left( \frac{f(t)}{t} \right)^\alpha \varphi(t) \, dt, \quad z \in U, \]  

(18)

which is regular in \( U \) and \( p(0) = p'(0) - 1 = 0 \).

We have

\[ \frac{zp''(z)}{p'(z)} = \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) + \frac{z\varphi'(z)}{\varphi(z)}, \quad z \in U, \]  

(19)

and hence, we obtain

\[ 1 - \frac{|z|^{2\alpha}}{a} \left| \frac{zp''(z)}{p'(z)} \right| \leq 1 - \frac{|z|^{2\alpha}}{a} \left[ |\alpha| \left| \frac{zf'(z)}{f(z)} - 1 \right| + \left| \frac{z\varphi'(z)}{\varphi(z)} \right| \right], \]  

(20)

for all \( z \in U \).

Applying Lemma 2, from (11) and (12) we get

\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq M|z|^n, \quad z \in U, \]  

(21)

\[ \left| \frac{z\varphi'(z)}{\varphi(z)} \right| \leq L|z|^n, \quad z \in U. \]  

(22)

From (20) and (21), (22) we obtain

\[ 1 - \frac{|z|^{2\alpha}}{a} \left| \frac{zp''(z)}{p'(z)} \right| \leq 1 - \frac{|z|^{2\alpha}}{a} |z|^n(|\alpha|M + L), \quad z \in U. \]  

(23)

We consider the function \( Q : [0,1] \rightarrow \mathbb{R}, Q(x) = \frac{(1-x^{2\alpha})x^n}{a}, \) where \( x = |z|, \) \( x \in [0,1]. \)

We have
\[
\max_{x \in [0,1]} Q(x) = \frac{2n^{\frac{n}{2}}}{(2a + n)^{\frac{2n+2}{2a}}} \text{, } n \in \mathbb{N} - \{0\}.
\] 

(24)

By (13), (24) and (23) we obtain

\[
1 - |z|^{2a} \frac{z p''(z)}{p'(z)} \leq 1,
\]

(25)

for all \( z \in U \).

Now, from (25) and Lemma 1, it results that

\[
G_{\alpha, \delta} \in \mathcal{S}, \quad G_{\alpha, \delta}(z) = z + b_2 z + b_3 z^2 + \cdots,
\]

(26)

hence, for \( \alpha + \delta = \beta + \gamma \), from (16) and (26) we have

\[
I_{\alpha, \beta, \gamma, \delta}(f)(z) = \frac{1}{\phi^{\frac{1}{\beta}}(z)} z \left(1 + b_2 z + b_3 z^2 + \cdots\right)^{\frac{\beta + \gamma}{\beta}}, \quad z \in U
\]

(27)

and

\[
z^{\frac{\alpha}{\beta + \gamma}} \phi^{\frac{1}{\beta + \gamma}}(z) I_{\alpha, \beta, \gamma, \delta}^{\frac{\beta}{\beta + \gamma}}(f)(z) \in \mathcal{S}.
\]

(28)

**Remark 1.** From Theorem 3, for \( \phi(z) = 1 \) and \( \gamma = 0 \) we have \( I_{\alpha, \beta, 0, \delta}(f)(z) \) belongs to class \( \mathcal{S} \).

**Corollary 4.** Let \( \alpha, \beta, \gamma, \delta \) be complex numbers, \( \alpha = \beta = e^{i\sigma}, \sigma \in \mathbb{R}, \delta = \gamma, a = \text{Re}(e^{i\sigma} + \gamma) > 0 \) and the functions \( \phi(z) = \varphi(z) = 1, z \in U \) and the function \( f \in A, f(z) = z + a_2 z^2 + \cdots, M \) positive real number.

If

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < M, \quad z \in U,
\]

(29)

\[
M \leq \frac{(2a + 1)^{\frac{2a+1}{2a}}}{2},
\]

(30)

then

\[
T_{\gamma, \sigma}(z) = z(1 + b_2 z + \cdots) e^{i\sigma + \gamma}, \quad z \in U
\]

(31)

and \( T_{0, \sigma}(z) \) belongs to class \( \mathcal{S} \).

**Proof.** Using (20) and Theorem 3 for \( n = 1 \), we obtain Corollary 4.
Corollary 5. Let \( \beta \) be a complex number, \( \Re \beta > 0 \), the functions \( \phi(z) = \varphi(z) = 1 \), \( z \in \mathcal{U} \) and the function \( f \in A \), \( f(z) = z + a_2 z^2 + \cdots \), \( M \) positive real number. If

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < M, \quad z \in \mathcal{U},
\]

then the integral operator Miller-Mocanu, \( J_\beta \), belongs to class \( \mathcal{S} \),

\[
J_\beta(z) = z + b_2 z^2 + \cdots, \quad z \in \mathcal{U}.
\]

Proof. We have \( \gamma = 0, \delta = 0, \alpha = \beta, \phi(z) = \varphi(z) = 1 \) and \( a = \Re \beta > 0 \). Applying Theorem 3 and using (19), we obtain Corollary 5.

Virgil Pescar
Department of Mathematics and Computer Science,
Faculty of Mathematics and Computer Science,
Transilvania University of Brașov,
500091, Brașov, Romania,
email: virgelpescar@unitbv.ro