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MULTIPLE SOLUTIONS FOR $P(X)$-LAPLACIAN-LIKE PROBLEMS WITH NEUMANN CONDITION

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Abstract. In this paper we investigate the existence of at least three weak solutions for the Neumann problem, originated from a capillary phenomena,

$$\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)+a(x)|u|^{p(x)-2}u= & \\ \lambda f(x,u)+\mu g(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, ν is the outer unit normal to $\partial\Omega$, $\lambda > 0$, $\mu \geq 0$, $a \in L^\infty(\Omega)$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions and $p \in C^0(\overline{\Omega})$. The approach is based on variational methods and critical point theory.

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1. Introduction

The study of differential and partial differential equations with variable exponent has been received considerable attention in recent years. This is partly due to their frequent appearance in applications such as the modeling of elastic mechanics [27], thermorheologic and and electro-rheological fluids [1, 3, 23] and image processing [11] and mathematical biology [18].

In this paper we shall discuss the existence of at least three weak solutions of the $p(x)$-Laplacian-like problem, originated from a capillary phenomena,

$$\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)+a(x)|u|^{p(x)-2}u= & \\ \lambda f(x,u)+\mu g(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, ν is the outer unit normal to $\partial\Omega$, $\lambda > 0$, $\mu \geq 0$, $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a \geq 0$, $f, g : \Omega\times\mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions and $p \in C^0(\overline{\Omega})$ satisfies the condition

$$N < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty.$$

Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e., the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e., the attractive force between the molecules of the liquid. The study of capillary phenomenon has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as motion of drops, bubbles and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems. In the context of the study of capillarity phenomena, many results have been obtained, for instance [2, 4, 9, 12, 17, 21, 22, 28]. For example, Obersnel and Omari in [22] studied the existence of positive solutions of the parametric problem

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f(t,u) \quad \text{in}\,\Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open subset with sufficiently smooth boundary $\partial\Omega$ and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function whose potential satisfies a suitable oscillating behaviour at zero. Rodrigues in [21], by using Mountain Pass lemma (see [10]) and Fountain theorem (see Theorem 3.6 in [25]), established the existence of non-trivial solutions for problem

$$\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(x,u), \quad x \in \Omega, \\ u = 0, \qquad x \in \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1, λ is a positive parameter, $p \in C(\overline{\Omega})$ and f is a Carathéodory function. Avci in [2] has considered the existence and multiplicity of solutions for nonlinear elliptic problem for the $p(x)$-Laplacian-like operators originated from a capillary phenomena. Zhou in [28], in view of the variational approach, discussed the nonlinear eigenvalue problems for $p(x)$-Laplacian-like operators, originated from a capillary phenomenon, and under some suitable conditions proved the existence of nontrivial solutions of the system for every parameter $\lambda > 0$. In [9] the authors, using a Fredholm-type result for a couple of nonlinear operators and the theory of variable exponent Sobolev spaces, obtained weak solutions for a class nonlinear elliptic problems for the $p(x)$-Laplacian-like operators under no-flux boundary conditions.

Problems like (1), (2) and (3) play, as is well known, a role in differential geometry and in the theory of relativity.

In the present paper, employing two kinds of three critical points theorems obtained in [8] and [5] which we recall in the next section (Theorems 1 and 2) we ensure the existence of exact collocations of the parameters λ and μ for which the problems (1) possesses at least three weak solutions.

The plan of the paper is as follows. In the next Section, we introduce our abstract framework. In the last Section, we discuss the existence of three weak solutions for the problem (1).

2. Preliminaries

Our main tools are two three-critical-point theorems that we recall here in convenient forms. The first one has been obtained in [8] and it is a more precise version of Theorem 3.2 of [5]. The second one has been established in [5].

Theorem 1 ([8, Theorem 2.6]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that

(i) $\sup_{\Phi(x)\leq r} \Psi(x) < r\Psi(\bar{x})/\Phi(\bar{x})$,

(ii) for each λ in

$$\Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\right[,$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2 ([5, Corollary 3.1]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose*

Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0\,.$$

Assume that there exist two positive constants $r_1, r_2 > 0$ and $\bar{x} \in X$, with $2r_1 < \Phi(\bar{x}) < \frac{r_2}{2}$, such that

(j) $\frac{\sup_{\Phi(x)<r_1} \Psi(x)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(jj) $\frac{\sup_{\Phi(x)<r_2} \Psi(x)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(jjj) for each λ in

$$\Lambda^*_{r_1,r_2} := \Big]\frac{3}{2}\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \min\Big\{\frac{r_1}{\sup_{\Phi(x)<r_1}\Psi(x)}, \frac{r_2}{2\sup_{\Phi(x)<r_2}\Psi(x)}\Big\}\Big[$$

and for every $x_1, x_2 \in X$, which are local minima for the functional $\Phi - \lambda\Psi$, and such that $I(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, one has $\inf_{t\in[0,1]} \Psi(tx_1+(1-t)x_2) \geq 0$.

Then, for each $\lambda \in \Lambda^*_{r_1,r_2}$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(-\infty, r_2)$.

We also refer the interested reader to the papers [6, 7, 13, 19] in which Theorems 1 and 2 have been successfully employed to ensure the existence of at least three solutions for boundary value problems.

For the reader's convenience, we state some basic properties of variable exponent Sobolev spaces and introduce some notations. For more details, we refer the reader to [14, 15, 16, 20, 23, 24]. Set

$$C_+(\Omega) := \big\{h \in C(\overline{\Omega}) : h(x) > 1,\ \forall x \in \overline{\Omega}\big\}.$$

For $p \in C_+(\Omega)$, define

$$L^{p(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\}.$$

We can introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{\beta > 0 : \int_\Omega \left|\frac{u(x)}{\beta}\right|^{p(x)} dx \leq 1\right\}.$$

The space $(L^{p(x)}(\Omega), |u|_{p(x)})$ is a Banach space called a variable exponent Lebesgue space. Define the Sobolev space with variable exponent

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

$W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space (see [14]).

When $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a \geq 0$, we define

$$L^{p(x)}_{a(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega a(x)|u(x)|^{p(x)}dx < +\infty\right\}$$

with the norm

$$|u|_{p(x),a(x)} = \inf\left\{\beta > 0 : \int_\Omega a(x)\left|\frac{u(x)}{\beta}\right|^{p(x)} dx \leq 1\right\}.$$

For any $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_a := \inf\left\{\beta > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\beta}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\beta}\right|^{p(x)}\right)dx \leq 1\right\}.$$

Then, it is easy to see that $\|u\|_a$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{1,p(x)}$. In the following, we will use $\|\cdot\|_a$ instead of $\|\cdot\|_{1,p(x)}$ on $X = W^{1,p(x)}(\Omega)$.

As pointed out in [15] and [20], X is continuously embedded in $W^{1,p^-}(\Omega)$ and, since $p^- > N$, $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$. Thus, X is compactly embedded in $C^0(\bar{\Omega})$. So, in particular, there exists a positive constant $k > 0$ such that

$$\|u\|_{C^0(\bar{\Omega})} \leq k\|u\|_a \tag{4}$$

for each $u \in X$. When Ω is convex, an explicit upper bound for the constant k is

$$k \leq 2^{\frac{p^--1}{p^-}} \max\left\{\left(\frac{1}{\|a\|_1}\right)^{\frac{1}{p^-}}, \frac{\sigma}{N^{\frac{1}{p^-}}}\left(\frac{p^- - 1}{p^- - N}|\Omega|\right)^{\frac{p^--1}{p^-}} \frac{\|a\|_\infty}{\|a\|_1}\right\}(1 + |\Omega|),$$

where $\sigma = \operatorname{diam}(\Omega)$ and $|\Omega|$ is the Lebesgue measure of Ω, $\|a\|_1 = \int_\Omega a(x)dx$ and $\|a\|_\infty = \sup_{x\in\Omega} a(x)$.

Lemma 3 ([15]). *Set $\rho(u) = \int_\Omega(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)})dx$. For $u \in X$ we have* *(i)* $\|u\|_a < (=;>)1 \Leftrightarrow \rho(u) < (=;>)1$,

(ii) $\|u\|_a < 1 \Rightarrow \|u\|_a^{p^+} \leq \rho(u) \leq \|u\|_a^{p^-}$,

(iii) $\|u\|_a > 1 \Rightarrow \|u\|_a^{p^-} \leq \rho(u) \leq \|u\|_a^{p^+}$.

We introduce the functions $F, G : \Omega \times \mathbb{R} \to \mathbb{R}$ corresponding respectively to the functions f and g, as follows

$$F(x,t) := \int_0^t f(x,\xi)\, d\xi$$

and

$$G(x,t) := \int_0^t g(x,\xi)\, d\xi$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

Moreover, set $G^c := \int_\Omega \sup_{|t|\leq c} G(x,t)\, dx$ for every $c > 0$ and $G_d := \inf_{\Omega\times[0,d]} G$ for every $d > 0$. If g is sign-changing, then $G^c \geq 0$ and $G_d \leq 0$.

Consider the following functional

$$\Phi(u) := \int_\Omega \frac{1}{p(x)}\left(|\nabla u(x)|^{p(x)} + \sqrt{1 + |\nabla u(x)|^{2p(x)}} + a(x)|u(x)|^{p(x)}\right) dx, \quad \forall u \in X.$$

Similar arguments as in [21] show that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega \left(|\nabla u(x)|^{p(x)-2}\nabla u(x) + \frac{|\nabla u(x)|^{2p(x)-2}\nabla u(x)}{\sqrt{1+|\nabla u(x)|^{2p(x)}}}\right)\nabla v(x)\, dx$$
$$+ \int_\Omega a(x)|u(x)|^{p(x)-2}u(x)\, v(x)\, dx$$

for every $v \in X$.

Proposition 1 ([21])**.** *The functional $\Phi : X \to \mathbb{R}$ is convex and the mapping $\Phi' : X \to X^*$ is a strictly monotone and bounded homeomorphism.*

We say that a function $u \in X$ is a *weak solution* of problem (1) if

$$\int_\Omega \left(|\nabla u(x)|^{p(x)-2}\nabla u(x) + \frac{|\nabla u(x)|^{2p(x)-2}\nabla u(x)}{\sqrt{1+|\nabla u(x)|^{2p(x)}}}\right)\nabla v(x) dx +$$
$$\int_\Omega a(x)|u(x)|^{p(x)-2}u(x)\, v(x)\, dx - \lambda \int_\Omega f(x,u(x))v(x)\, dx - \mu \int_\Omega g(x,u(x))v(x)\, dx = 0$$

holds for all $v \in X$.

3. Main results

Fixing $d \geq 1$ and $c \geq k$ such that

$$\frac{(|\Omega| + \|a\|_1)d^{p^+}}{p^- \int_\Omega F(x,d)dx} < \frac{\left(\frac{c}{k}\right)^{p^-}}{p^+ \int_\Omega \max_{|t|\leq c} F(x,t)dx}$$

and picking

$$\lambda \in \Lambda_1 := \left]\frac{(|\Omega| + \|a\|_1)d^{p^+}}{p^- \int_\Omega F(x,d)dx}, \frac{\left(\frac{c}{k}\right)^{p^-}}{p^+ \int_\Omega \max_{|t|\leq c} F(x,t)dx}\right[, \tag{5}$$

put

$$\delta_1 := \min\left\{\frac{c^{p^-} - \lambda p^+ k^{p^-} \int_\Omega \max_{|t|\leq c} F(x,t)dx}{p^+ k^{p^-} G^c}, \right.$$
$$\left. \left|\frac{(|\Omega| + \|a\|_1)d^{p^+} - \lambda p^- \int_\Omega F(x,d)\,dx}{p^-|\Omega| G_d}\right|\right\} \tag{6}$$

and

$$\overline{\delta}_1 := \min\left\{\delta_1, \frac{1}{\max\left\{0, p^+ k^{p^-}|\Omega| \limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} G(x,\xi)}{\xi^{p^-}}\right\}}\right\}, \tag{7}$$

where that for instance $\overline{\delta}_1 = +\infty$ when

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} G(x,\xi)}{\xi^{p^-}} \leq 0,$$

and $G_d = G^c = 0$.

Now, we formulate our main result as follows.

Theorem 4. *Suppose that there exist $d \geq 1$ and $c \geq k$ with*

$$d^{p^-}\|a\|_1 > \left(\frac{c}{k}\right)^{p^-}, \tag{8}$$

such that

(A1) $\frac{\int_\Omega \max_{|t|\leq c} F(x,t)dx}{\left(\frac{c}{k}\right)^{p^-}} < \frac{p^- \int_\Omega F(x,d)dx}{p^+ d^{p^+}(|\Omega|+\|a\|_1)}$;

(A2) $\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{\xi^{p^-}} \leq 0$.

Then, for every $\lambda \in \Lambda_1$, *where* Λ_1 *is given by (5), and for every* L^1*-Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying the condition*

$$\limsup_{|\xi| \to +\infty} \frac{\sup_{x\in\Omega} G(x,\xi)}{\xi^{p^-}} < +\infty, \tag{9}$$

there exists $\overline{\delta}_1 > 0$ *given by (7) such that, for each* $\mu \in [0, \overline{\delta}_1[$, *the problem (1) admits at least three distinct weak solutions in* X.

Proof. Fix λ, μ and g as in the conclusion. For each $u \in X$, we let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \int_\Omega \frac{1}{p(x)} \Big(|\nabla u(x)|^{p(x)} + \sqrt{1 + |\nabla u(x)|^{2p(x)}} + a(x)|u(x)|^{p(x)} \Big) dx,$$

$$\Psi(u) := \int_\Omega \Big(F(x, u(x))dx + \frac{\mu}{\lambda} G(x, u(x)) \Big) dx$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u).$$

Note that the weak solutions of (1) are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 1. Indeed, we have already pointed out that Φ is C^1 on X and sequentially weakly lower semi-continuous. Furthermore, Proposition 1 gives that $\Phi' : X \to X^*$ admits a continuous inverse, and Lemma 3 follows that Φ is coercive. On the other hand, it is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_\Omega \Big(f(x, u(x)) + \frac{\mu}{\lambda} g(x, u(x)) \Big) v(x)\, dx$$

for any $v \in X$ as well as it is sequentially weakly upper semicontinuous. Furthermore $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then u_n converges uniformly to u on Ω as $n \to \infty$; see [26]. Since f, g are continuous functions in $\mathbb{R}$ for every $x \in \Omega$, so

$$f(x, u_n) + \frac{\mu}{\lambda} g(x, u_n) \to f(x, u) + \frac{\mu}{\lambda} g(x, u)$$

as $n \to \infty$. Hence $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we have proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [26]. Choose $w(x) := d$ for all $x \in \Omega$ and

$$r := \frac{1}{p^+} \big(\frac{c}{k}\big)^{p^-}.$$

Clearly, $w \in X$ and from the condition (8) one has

$$\Phi(w) = \int_\Omega \Big[\frac{1}{p(x)} + \frac{a(x)}{p(x)} d^{p(x)}\Big] dx \geq \frac{1}{p^+} d^{p^-} \|a\|_1 > r.$$

Also, we have

$$\begin{aligned}
\Psi(w) &= \int_\Omega \Big[F(x,d) + \frac{\mu}{\lambda} G(x,d)\Big] dx \\
&\geq \int_\Omega F(x,d)dx + \frac{\mu}{\lambda}|\Omega| \inf_{\Omega\times[0,d]} G \\
&= \int_\Omega F(x,d)dx + \frac{\mu}{\lambda}|\Omega| G_d.
\end{aligned}$$

By Lemma 3 and the fact $\max\{r^{1/p^-}, r^{1/p^+}\} = r^{1/p^-}$, we deduce

$$\{u \in X : \Phi(u) < r\} \subseteq \Big\{u \in X : \|u\|_a < r^{1/p^-}\Big\} = \Big\{u \in X : \|u\|_a < \frac{c}{k}\Big\}.$$

Moreover, due to (4), we have

$$|u(x)| \leq \|u\|_\infty \leq k\|u\|_a \leq c, \quad \forall x \in \overline{\Omega}.$$

Hence,

$$\Big\{u \in X : \|u\|_a < \frac{c}{k}\Big\} \subseteq \{u \in X : \|u\|_\infty \leq c\}.$$

Therefore,

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{r} &\leq \sup_{u\in\Phi^{-1}(-\infty,r]} \int_\Omega \Big(F(x,u(x)) + \frac{\mu}{\lambda} G(x,u(x))\Big) dx \\
&\leq \frac{\displaystyle\int_\Omega \sup_{|t|\leq c} F(x,t)dx + \frac{\mu}{\lambda} G^c}{\frac{1}{p^+}\left(\frac{c}{k}\right)^{p^-}}.
\end{aligned}$$

From this, if $G^c = 0$, it is clear that we get

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{1}{\lambda}, \tag{10}$$

while, if $G^c > 0$, it turns out to be true bearing in mind that

$$\mu < \frac{c^{p^-} - \lambda p^+ k^{p^-} \int_\Omega \sup_{|t|\leq c} F(x,t)dx}{p^+ k^{p^-} G^c}.$$

On the other hand, taking into account that

$$0 < \Phi(w) \leq \frac{1}{p^-}(|\Omega| + \|a\|_1)d^{p^+},$$

we have

$$\frac{\Psi(w)}{\Phi(w)} \quad \leq \quad \frac{\displaystyle\int_\Omega F(x,d)dx + \frac{\mu}{\lambda}|\Omega|G_d}{\frac{1}{p^-}(|\Omega| + \|a\|_1)d^{p^+}}.$$

Hence, if $G_d \geq 0$, one has

$$\frac{\Psi(w)}{\Phi(w)} > \frac{1}{\lambda}, \tag{11}$$

while, if $G_d < 0$, it holds since

$$\mu < \frac{(|\Omega| + \|a\|_1) - \lambda p^- \int_\Omega F(x,d)dx}{p^-|\Omega|G_d}.$$

Therefore, from (10) and (11), condition (i) of Theorem 1 is fulfilled. Finally, from (9), since $\mu < \overline{\delta}_1$, we can fix $l > 0$ such that $\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} G(x,\xi)}{\xi^{p^-}} < l$ and $\mu l < \frac{1}{p^+k^{p^-}|\Omega|}$. Therefore, there exists a function $h \in L^1(\Omega)$ such that

$$G(x,t) \leq lt^{p^-} + h(x)$$

for every $x \in \Omega$ and $t \in \mathbb{R}$. Now, fix $0 < \varepsilon < \frac{1}{p^+k^{p^-}|\Omega|\lambda} - \frac{\mu l}{\lambda}$. From (A2) there is a function $h_\varepsilon \in L^1(\Omega)$ such that

$$F(x,t) \leq \varepsilon t^{p^-} + h_\varepsilon(x)$$

for every $x \in \Omega$ and $t \in \mathbb{R}$. Taking (4) into account, it follows that, for each $u \in X$,

$$\begin{aligned} I_\lambda(u) = \Phi(u) - \lambda\Psi(u) &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \int_\Omega \Big(F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))\Big)dx \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \lambda\varepsilon\int_\Omega (u(x))^{p^-}dx - \lambda\|h_\varepsilon\|_{L^1(\Omega)} - \mu l\int_\Omega (u(x))^{p^-}dx - \mu\|h\|_{L^1(\Omega)} \\ &\geq \Big(\frac{1}{p^+} - \lambda k^{p^-}|\Omega|\varepsilon - \mu k^{p^-}|\Omega|l\Big)\|u\|_a^{p^-} - \lambda\|h_\varepsilon\|_{L^1(\Omega)} - \mu\|h\|_{L^1(\Omega)}, \end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty,$$

which means the functional I_λ is coercive and the condition (ii) of Theorem 1 is verified. Since from (10) and (11),

$$\lambda \in \Lambda_1 \subseteq \left] \frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)} \right[,$$

Theorem 1 (with $\bar{x} = w$) ensures the existence of at least three critical points for the functional I_λ in X, which are the weak solutions of the problem (1). This completes the proof.

Now, a variant of Theorem 4 in which no asymptotic condition on g is requested. In such a case f and g are supposed to be non-negative.

Fixing $d \geq 1$ and $c_1, c_2 > 0$ such that

$$\frac{3}{2}\frac{(|\Omega| + \|a\|_1)d^{p^+}}{p^- \int_\Omega F(x,d)dx} < \frac{1}{p^+ k^{p^-}} \min\left\{ \frac{c_1^{p^-}}{\int_\Omega \sup_{|t|\leq c_1} F(x,t)dx}, \frac{c_2^{p^-}}{2\int_\Omega \sup_{|t|\leq c_2} F(x,t)dx} \right\},$$

and picking

$$\lambda \in \Lambda_2 := \left] \frac{3}{2}\frac{(|\Omega| + \|a\|_1)d^{p^+}}{p^- \int_\Omega F(x,d)dx}, \right. \left. \frac{1}{p^+ k^{p^-}} \min\left\{ \frac{c_1^{p^-}}{\int_\Omega \sup_{|t|\leq c_1} F(x,t)dx}, \frac{c_2^{p^-}}{2\int_\Omega \sup_{|t|\leq c_2} F(x,t)dx} \right\} \right[, \tag{12}$$

put

$$\delta_2 := \min\left\{ \frac{c_1^{p^-} - \lambda p^+ k^{p^-} \int_\Omega \sup_{|t|\leq c_1} F(x,t)dx}{p^+ k^{p^-} G^{c_1}}, \frac{c_2^{p^-} - 2\lambda p^+ k^{p^-} \int_\Omega \sup_{|t|\leq c_2} F(x,t)dx}{2p^+ k^{p^-} G^{c_2}} \right\}. \tag{13}$$

With the above notations we have the following multiplicity result.

Theorem 5. *Suppose that there exist $d \geq 1$ and two constants c_1, c_2 with $\min\{c_1, c_2\} \geq k$ and*

$$2\Big(\frac{c_1}{k}\Big)^{p^-} < d^{p^-}\|a\|_1, \qquad (|\Omega| + \|a\|_1)d^{p^+} < \frac{p^-}{2p^+}\Big(\frac{c_2}{k}\Big)^{p^-},$$

such that

(B1) $f(x,\xi) \geq 0$ for all $(x,\xi) \in \Omega \times \mathbb{R}$;

(B2) $\max\left\{\frac{\int_\Omega \sup_{|t|\le c_1} F(x,t)dx}{\left(\frac{c_1}{k}\right)^{p^-}}, \frac{2\int_\Omega \sup_{|t|\le c_2} F(x,t)dx}{\left(\frac{c_2}{k}\right)^{p^-}}\right\} < \frac{2}{3}\frac{p^-\int_\Omega F(x,d)dx}{p^+(|\Omega|+\|a\|_1)d^{p^+}}$.

Then, for every $\lambda \in \Lambda_2$ is given by (12), and for non-negative L^1-Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ there exists $\delta_2 > 0$ given by (13) such that, for each $\mu \in [0, \delta_2[$, the problem (1) admits at least three distinct weak solutions u_i, $i = 1, 2, 3$, such that

$$0 \le u_i(x) < c_2, \qquad \forall x \in \Omega,\ i = 1, 2, 3.$$

Proof. Fix λ, μ and g as in the conclusion and take X, Φ, Ψ and I_λ as in the proof of Theorem 4. We observe that the regularity assumptions of Theorem 2 on Φ and Ψ are satisfied. Then, our aim is to verify (j) and (jj). Put $w(x) := d$ for all $x \in \Omega$, $r_1 := \frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}$ and $r_2 := \frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}$. Therefore, since

$$\begin{aligned}\frac{1}{p^+}\|a\|_1 d^{p^-} &\le \int_\Omega \frac{a(x)}{p(x)} d^{p(x)}dx \le \Phi(w) = \int_\Omega \Big(\frac{1}{p(x)} + \frac{a(x)}{p(x)} d^{p(x)}\Big)dx \\ &\le \frac{1}{p^-}(|\Omega| + \|a\|_1)d^{p^+},\end{aligned}$$

by using the conditions

$$2\Big(\frac{c_1}{k}\Big)^{p^-} < d^{p^-}\|a\|_1, \qquad (|\Omega| + \|a\|_1)d^{p^+} < \frac{p^-}{2p^+}\Big(\frac{c_2}{k}\Big)^{p^-},$$

one has $2r_1 < \Phi(w) < \frac{r_2}{2}$. Since $\mu < \delta_2$ and $G_d \ge 0$, one has

$$\begin{aligned}\frac{1}{r_1}\sup_{\Phi(u)<r_1}\Psi(u) &= \frac{1}{r_1}\sup_{\Phi(u)<r_1}\int_\Omega \Big(F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))\Big)\,dx \\ &\le \frac{\int_\Omega \sup_{|t|\le c_1} F(x,t)dx + \frac{\mu}{\lambda}G^{c_1}}{\frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}} \\ &< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_\Omega F(x,d)dx + \frac{\mu}{\lambda}|\Omega|G_d}{\frac{1}{p^-}(|\Omega| + \|a\|_1)d^{p^+}} \\ &\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},\end{aligned}$$

and

$$\begin{aligned}\frac{2}{r_2}\sup_{\Phi(u)<r_2}\Psi(u) &= \frac{2}{r_2}\sup_{\Phi(u)<r_2}\int_\Omega\Big(F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))\Big)\,dx\\ &\le \frac{2\int_\Omega\sup_{|t|\le c_2}F(x,t)dx+2\frac{\mu}{\lambda}G^{c_2}}{\frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}}\\ &< \frac{1}{\lambda}<\frac{2}{3}\frac{\int_\Omega F(x,d)dx+\frac{\mu}{\lambda}|\Omega|G_d}{\frac{1}{p^-}(|\Omega|+\|a\|_1)d^{p^+}}\\ &\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

Therefore, conditions (j) and (jj) of Theorem 2 are satisfied. Finally, we verify that I_λ satisfies the assumption (jjj) of Theorem 2. Let u_1 and u_2 be two local minima for I_λ. Then, u_1 and u_2 are critical points for I_λ, and so, they are weak solutions for the problem (1). We claim that the weak solutions obtained are non-negative. Indeed, let $\bar{u}\in X$ be one (non-trivial) weak solution of the problem (1), then one has

$$\int_\Omega\left(|\nabla\bar{u}(x)|^{p(x)-2}\nabla\bar{u}(x)+\frac{|\nabla\bar{u}(x)|^{2p(x)-2}\nabla\bar{u}(x)}{\sqrt{1+|\nabla\bar{u}(x)|^{2p(x)}}}\right)\nabla v(x)dx+$$
$$\int_\Omega a(x)|\bar{u}|^{p(x)-2}\bar{u}(x)\,v(x)\,dx=\lambda\int_\Omega f(x,\bar{u}(x))v(x)\,dx+\mu\int_\Omega g(x,\bar{u}(x))v(x)\,dx$$

for all $v\in X$. Arguing by a contradiction and setting

$$\Omega^-:=\{x\in\Omega:\bar{u}(x)<0\},$$

one has $\Omega^-\neq\emptyset$. Put $\bar{v}:=\min\{\bar{u},0\}$, one has $\bar{v}\in X$. So, taking into account that $\bar{u}$ is a weak solution and by choosing $v=\bar{v}$, from our sign assumptions on the data, we have

$$\begin{aligned}&\int_{\Omega^-}\left(|\nabla\bar{u}(x)|^{p(x)}+\frac{|\nabla\bar{u}(x)|^{2p(x)}}{\sqrt{1+|\nabla\bar{u}(x)|^{2p(x)}}}\right)dx+\int_{\Omega^-}a(x)|\bar{u}(x)|^{p(x)}\,dx\\ &\qquad=\lambda\int_{\Omega^-}f(x,\bar{u}(x))\bar{u}(x)\,dx+\mu\int_{\Omega^-}g(x,\bar{u}(x))\bar{u}(x)\,dx\le 0.\end{aligned}$$

we observe that $\|\bar{u}\|_{W^{1,p(x)}(\Omega^-)}=0$ which is absurd. Then, we obtain $u_1(x)\ge 0$ and $u_2(x)\ge 0$ for all $x\in\Omega$. So, one has $\Psi(su_1+(1-s)u_2)\ge 0$ for all $s\in[0,1]$. Therefore, also (jjj) holds. From Theorem 2 the functional I_λ has at least three distinct critical points which are weak solutions of (1). This completes the proof.

A special case of Theorem 4 is the following theorem.

Theorem 6. *Let $p(x) = p > N$ for every $x \in \Omega$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) := \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that $F(d) > 0$ for some $d \geq 1$ and*

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^p} = \limsup_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^p} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|\xi| \to +\infty} \frac{\int_0^t g(s)ds}{t^p} < +\infty,$$

there exists $\delta^* > 0$ such that for each $\mu \in [0, \delta^*[$, the problem

$$\begin{cases} -\mathrm{div}\left(\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = \lambda f(u) + \mu g(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least three distinct weak solutions in X.

Proof. Fix $\lambda > \lambda^* := \frac{2d^p}{pF(d)}$ for some $d \geq 1$ such that $F(d) > 0$. Since

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^p} = 0,$$

there is a sequence $\{c_n\} \subset]0, +\infty[$ such that $\lim_{n \to +\infty} c_n = 0$ and

$$\lim_{n \to +\infty} \frac{\max_{|\xi| \leq c_n} F(\xi)}{c_n^p} = 0.$$

Indeed, one has

$$\lim_{n \to +\infty} \frac{\max_{|\xi| \leq c_n} F(\xi)}{c_n^p} = \lim_{n \to +\infty} \frac{F(\xi_{c_n})}{\xi_{c_n}^p} \frac{\xi_{c_n}^p}{c_n^p} = 0,$$

where $F(\xi_{c_n}) := \max_{|\xi| \leq c_n} F(\xi)$. Therefore, there exists $\bar{c} \geq k$ such that

$$\frac{\max_{|\xi| \leq \bar{m}} F(\xi)}{\bar{m}^p} < \min\left\{\frac{F(d)}{2(cd)^p|\Omega|}, \frac{1}{pc^p\lambda|\Omega|}\right\}$$

and $\bar{m} < cd|\Omega|^{1/p}$. Hence, the conclusion follows from Theorem 4.

The following result is a consequence of Theorem 5.

Theorem 7. *Let* $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a non-negative continuous function such that*

$$\liminf_{t \to 0^+} \frac{f(t)}{t^2} = 0.$$

and

$$\int_0^{128} f(\xi) d\xi < \frac{2^{11}}{3(1+\pi)^3} \int_0^2 f(\xi)\, d\xi.$$

Then, for every

$$\lambda \in \left] \frac{8}{\int_0^2 f(\xi)\, d\xi}, \frac{4^7}{3(1+\pi)^3 \int_0^{128} f(\xi) d\xi} \right[$$

and for every non-negative continuous $g : \mathbb{R} \to \mathbb{R}$, *there exists* $\delta^* > 0$ *such that for each* $\mu \in [0, \delta^*[$, *the problem*

$$\begin{cases} -\mathrm{div}\left(\left(1 + \frac{|\nabla u|^3}{\sqrt{1+|\nabla u|^6}}\right)|\nabla u|\nabla u\right) + |u|u = \lambda f(u) + \mu g(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least three distinct weak solutions in X.

Proof. Our aim is to apply Theorem 5 by choosing $c_2 = 128$ and $d = 2$. Therefore, taking into account that $k = 4(1+\pi)$, one has

$$\frac{3}{2} \frac{(|\Omega| + \|a\|_1) d^{p^+}}{p^- \int_\Omega F(x,d) dx} = \frac{8}{\int_0^2 f(\xi)\, d\xi}$$

and

$$\frac{1}{p^+ k^{p^-}} \frac{c_2^{p^-}}{2 \int_\Omega \sup_{|t| \le c_2} F(x,t) dx} = \frac{4^7}{3(1+\pi)^3 \int_0^{128} f(\xi) d\xi}.$$

Moreover, since $\lim_{t \to 0^+} \frac{f(t)}{t^2} = 0$, one has

$$\lim_{t \to 0^+} \frac{\int_0^t f(\xi) d\xi}{t^3} = 0.$$

Then, there exists a positive constant $c_1 < 4\sqrt[3]{4}(1+\pi)$ such that

$$\frac{\int_0^{c_1^3} f(\xi)\, d\xi}{c_1^3} < \frac{1}{3 \times 4^3 (1+\pi)^3} \int_0^2 f(\xi)\, d\xi,$$

and

$$\frac{c_1^3}{\int_0^{c_1^3} f(\xi)\, d\xi} > \frac{2^{20}}{\int_0^{128} f(\xi) d\xi}.$$

Hence, a simple computation shows that all assumptions of Theorem 5 are satisfied, and the conclusion follows.

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