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CERTAIN SUFFICIENT CONDITIONS FOR STARLIKE AND CONVEX FUNCTIONS

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Abstract. Using the technique of the differential subordination, we, here, obtain certain sufficient conditions for starlike, parabolic starlike, convex and uniformly convex functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{E} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore, Taylor's series expansion of $f \in \mathcal{A}$, is given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)),\ \ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for

all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha(0 \leq \alpha < 1)$ in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{E}.$$

Let $\mathcal{S}^*(\alpha)$ denote the class of starlike functions of order α. Write $\mathcal{S}^*(0) = \mathcal{S}^*$, the class of starlike functions.
A function $f \in \mathcal{A}$ is said to be convex of order $\alpha(0 \leq \alpha < 1)$ in $\mathbb{E}$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{E}.$$

Let the class of such functions be denoted by $\mathcal{K}(\alpha)$. Note that $\mathcal{K}(0) = \mathcal{K}$, the class of convex functions.
A function $f \in \mathcal{A}$ is said to be parabolic starlike in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{E}.$$

The class of parabolic starlike functions is denoted by $\mathcal{S_P}$. A function $f \in \mathcal{A}$ is said to be uniformly convex in $\mathbb{E}$ if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, z \in \mathbb{E}.$$

Let UCV denote the class of all such functions.
In 2003, Irmak et al. [4] studied the class $T_\lambda(\alpha)$ consisting of functions $f \in \mathcal{A}$ satisfying the following condition

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \prec 1 + (1-\alpha)z, 0 \leq \alpha < 1, z \in \mathbb{E},$$

and obtained certain conditions for $f \in \mathcal{A}$ to be a member of class $T_\lambda(\alpha)$ and consequently, they get some sufficient conditions for starlike and convex functions. The work of Irmak et al. ([4], [5], [6]) is the main source of motivation for the present paper.
Let $\mathcal{S}(\lambda, \alpha)$ denote the class of functions $f \in \mathcal{A}$ for which

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right) > \alpha, 0 \leq \lambda \leq 1, 0 \leq \alpha < 1, z \in \mathbb{E}. \quad (2)$$

Note that $\mathcal{S}(0,\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{S}(1,\alpha) = \mathcal{K}(\alpha)$.
Let $\mathcal{S}(\lambda)$ denote the class of functions $f \in \mathcal{A}$ for which

$$\Re\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right) > \left|\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - 1\right|, 0 \le \lambda \le 1, z \in \mathbb{E}. \quad (3)$$

Clearly, $\mathcal{S}(0)$ and $\mathcal{S}(1)$ are usual classes $\mathcal{S}_{\mathcal{P}}$ and UCV respectively. Define the parabolic domain Ω as under:

$$\Omega = \{u + iv : u > \sqrt{(u-1)^2 + v^2}\}.$$

Note that the condition (3) is equivalent to the condition that $\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right)$ take values in the parabolic domain Ω.
Ronning [2] and Ma and Minda [1] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad (4)$$

maps the unit disk $\mathbb{E}$ onto the parabolic domain Ω.
Therefore, equivalently condition (3) can be written as:

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec q(z)$$

where q(z) is given by (4).

In the present paper, we obtain sufficient conditions for a function $f \in \mathcal{A}$ to be a member of class $\mathcal{S}(\lambda,\alpha)$ and $\mathcal{S}(\lambda)$. As consequences of our main result, we obtain sufficient conditions for starlikeness, parabolic starlikeness, convexity and uniform convexity of analytic univalent functions.

2. Preliminaries

To prove our main results, we shall use the following lemma of Miller and Mocanu [3].

Lemma 1. *Let q be a univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q(z)$ and suppose that either*
(i) h is convex, or
(ii) Q is starlike.
In addition, assume that

(iii) $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ *for all* z *in* $\mathbb{E}$. *If* p *is analytic in* $\mathbb{E}$, *with* $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ *and*

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], z \in \mathbb{E},$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

3. Main Results

Theorem 2. *Let* $\beta \neq 0$ *be a complex number. Let* $q(z)$ *be a univalent function in* $\mathbb{E}$ *such that*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] > \max\left\{0, -\Re\left(\frac{q(z)}{\beta}\right)\right\} \tag{5}$$

If $f \in \mathcal{A}$ *satisfies*

$$(1-\beta)\left[\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right] + \beta\left[\frac{f'(z) + (1+2\lambda)zf''(z) + \lambda z^2 f''(z)}{f'(z) + \lambda z f''(z)}\right] \prec q(z) + \frac{\beta z q'(z)}{q(z)}, \tag{6}$$

then

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \prec q(z), z \in \mathbb{E},$$

where $0 \leq \lambda \leq 1$ *and* $q(z)$ *is the best dominant.*

Proof. On writing, $\dfrac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} = u(z)$, in (6), we obtain:

$$u(z) + \frac{\beta z u'(z)}{u(z)} \prec q(z) + \frac{\beta z q'(z)}{q(z)}.$$

Let us define the function θ and ϕ as follows:

$$\theta(w) = w$$

and

$$\phi(w) = \frac{\beta}{w}.$$

Clearly, θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Therefore,

$$Q(z) = \phi(q(z))zq'(z) = \frac{\beta z q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{\beta z q'(z)}{q(z)}.$$

On differentiating, we obtain $\dfrac{zQ'(z)}{Q(z)} = 1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{q(z)}$ and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{q(z)}{\beta}.$$

In view of the given conditions, we see that Q is starlike and $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$. Therefore, the proof, now follows from Lemma 1.

4. Applications:

Remark 1. *When we select the dominant* $q(z) = 1 + \dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ *in Theorem 2, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{q(z)}{\beta} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$+\frac{1}{\beta}\left(1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right).$$

Thus for positive real number β, we notice that q(z) satisfies the condition (5) in Theorem 2. Therefore, we immediately arrive at the following result.

Theorem 3. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left[\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right] + \beta\left[\frac{f'(z) + (1+2\lambda)zf''(z) + \lambda z^2 f''(z)}{f'(z) + \lambda z f''(z)}\right]$$

$$\prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \beta\left[\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right], z \in \mathbb{E},$$

then $f \in \mathcal{S}(\lambda), 0 \le \lambda \le 1$.

Setting $\lambda = 0$ in Theorem 3, we get the following result.

Corollary 4. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \beta\left[\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right], z \in \mathbb{E},$$

then $f \in \mathcal{S}_{\mathcal{P}}$.

Setting $\lambda = 1$ in Theorem 3, we get the following result.

Corollary 5. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left(1+\frac{zf''(z)}{f'(z)}\right) + \beta\left(\frac{f'(z)+3zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)$$

$$\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \beta\left[\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right], z \in \mathbb{E},$$

then $f \in UCV$.

Remark 2. *When we select the dominant $q(z) = e^z$ in Theorem 2, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = 1$$

and

$$1+\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{q(z)}{\beta} = 1+\frac{e^z}{\beta}.$$

Thus for positive real number β, we notice that q(z) satisfies the condition (5) in Theorem 2. Therefore, we immediately arrive at the following result.

Theorem 6. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left[\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right] + \beta\left[\frac{f'(z)+(1+2\lambda)zf''(z)+\lambda z^2f''(z)}{f'(z)+\lambda zf''(z)}\right] \prec e^z + \beta z,$$

then

$$\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec e^z, 0 \le \lambda \le 1, z \in \mathbb{E}.$$

Setting $\lambda = 0$ in Theorem 6, we get the following result.

Corollary 7. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec e^z + \beta z, z \in \mathbb{E},$$

then $f \in \mathcal{S}^*$.

Setting $\lambda = 1$ in Theorem 6, we obtain the following result.

Corollary 8. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left(1 + \frac{zf''(z)}{f'(z)}\right) + \beta\left(\frac{f'(z) + 3zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right) \prec e^z + \beta z, z \in \mathbb{E},$$

then $f \in \mathcal{K}$.

Remark 3. *When we select the dominant $q(z) = \dfrac{1 + (1-2\alpha)z}{1-z}; 0 \le \alpha < 1$ in Theorem 2, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1 + (1-2\alpha)z^2}{(1-z)(1 + (1-2\alpha)z)}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{q(z)}{\beta} = \frac{1 + (1-2\alpha)z^2}{(1-z)(1 + (1-2\alpha)z)} + \frac{1 + (1-2\alpha)z}{\beta(1-z)}.$$

Thus for positive real number β, we notice that q(z) satisfies the condition (5) in Theorem 2. Therefore, we immediately arrive at the following result.

Theorem 9. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left[\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)}\right] + \beta\left[\frac{f'(z) + (1+2\lambda)zf''(z) + \lambda z^2 f''(z)}{f'(z) + \lambda zf''(z)}\right]$$

$$\prec \frac{1 + (1-2\alpha)z}{1-z} + \frac{2\beta z(1-\alpha)}{(1-z)[1 + (1-2\alpha)z]}, z \in \mathbb{E},$$

then $f \in \mathcal{S}(\lambda, \alpha)$, where $0 \le \lambda \le 1, 0 \le \alpha < 1$.

Setting $\lambda = 0$ in Theorem 9, we get the following result.

Corollary 10. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+(1-2\alpha)z}{1-z}+\frac{2\beta z(1-\alpha)}{(1-z)[1+(1-2\alpha)z]}, 0 \leq \alpha < 1, z \in \mathbb{E}.$$

then $f \in \mathcal{S}^*(\alpha)$.

Setting $\lambda = 1$ in Theorem 9, we obtain the following result.

Corollary 11. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left(1+\frac{zf''(z)}{f'(z)}\right)+\beta\left(\frac{f'(z)+3zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)$$

$$\prec \frac{1+(1-2\alpha)z}{1-z}+\frac{2\beta z(1-\alpha)}{(1-z)[1+(1-2\alpha)z]}, 0 \leq \alpha < 1, z \in \mathbb{E}.$$

then $f \in \mathcal{K}(\alpha)$.

Remark 4. *When we select the dominant $q(z) = 1 + az; 0 \leq a < 1$ in Theorem 2, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1}{1+az}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{q(z)}{\beta}=\frac{1}{1+az}+\frac{1+az}{\beta}.$$

Thus for positive real number β, we notice that q(z) satisfies the condition (5) in Theorem 2. Therefore, we immediately arrive at the following result.

Theorem 12. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left[\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right]+\beta\left[\frac{f'(z)+(1+2\lambda)zf''(z)+\lambda z^2f''(z)}{f'(z)+\lambda zf''(z)}\right] \prec 1+az+\frac{\beta az}{1+az},$$

then

$$\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec 1+az, 0 \leq a < 1, 0 \leq \lambda \leq 1, z \in \mathbb{E}.$$

Remark 5. *When we select the dominant $q(z) = \dfrac{1+z}{1-z}$ in Theorem 2, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1+z^2}{1-z^2}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{q(z)}{\beta}=\frac{1+z^2}{1-z^2}+\frac{1+z}{\beta(1-z)}.$$

Thus for positive real number β, we notice that q(z) satisfies the condition (5) in Theorem 2. Therefore, we, immediately arrive at the following result.

Theorem 13. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left[\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}\right]+\beta\left[\frac{f'(z)+(1+2\lambda)zf''(z)+\lambda z^2 f''(z)}{f'(z)+\lambda z f''(z)}\right]\prec\frac{1+z}{1-z}+\frac{2\beta z}{1-z^2},$$

then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}\prec\frac{1+z}{1-z}, 0\le\lambda\le 1, z\in\mathbb{E}.$$

Remark 6. *When we select the dominant $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\le 1$ in Theorem 2, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1+z^2}{1-z^2}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{q(z)}{\beta}=\frac{1+z^2}{1-z^2}+\frac{1}{\beta}\left(\frac{1+z}{1-z}\right)^{\gamma}.$$

Thus for positive real number β, we notice that q(z) satisfies the condition (5) in Theorem 2. Therefore, we, immediately arrive at the following result.

Theorem 14. *Let β be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$(1-\beta)\left[\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}\right]+\beta\left[\frac{f'(z)+(1+2\lambda)zf''(z)+\lambda z^2 f''(z)}{f'(z)+\lambda z f''(z)}\right]\prec\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\beta\gamma z}{1-z^2},$$

then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}, 0\le\lambda\le 1, 0<\gamma\le 1, z\in\mathbb{E}.$$

Remark 7. *When we select the dominant $q(z)=\frac{\alpha'(1-z)}{\alpha'-z}$, $\alpha'>1$ in Theorem 2, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1}{1-z}+\frac{z}{\alpha'-z}$$

and
$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{q(z)}{\beta}=\frac{1}{1-z}+\frac{z}{\alpha'-z}+\frac{1}{\beta}\left(\frac{\alpha'(1-z)}{\alpha'-z}\right).$$
Thus for positive real number β, we notice that q(z) satisfies the condition (5) in Theorem 2. Therefore, we, immediately arrive at the following result.

Theorem 15. *Let β be a positive real number. If $f\in\mathcal{A}$ satisfies*
$$(1-\beta)\left[\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right]+\beta\left[\frac{f'(z)+(1+2\lambda)zf''(z)+\lambda z^2f''(z)}{f'(z)+\lambda zf''(z)}\right]$$
$$\prec\frac{\alpha'(1-z)}{\alpha'-z}+\frac{\beta(1-\alpha')z}{(1-z)(\alpha'-z)},$$
then
$$\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\prec\frac{\alpha'(1-z)}{\alpha'-z},\alpha'>1,0\leq\lambda\leq 1,z\in\mathbb{E}.$$

Remark 8. *Selecting $\lambda=0$ and $\lambda=1$ in above results, we get sufficient conditions for starlikeness and convexity for the function $f\in\mathcal{A}$ as discussed in Theorem 6 and Theorem 9.*

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