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COEFFICIENT ESTIMATES FOR NEW SUBCLASSES OF MEROMORPHICALLY BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we have introduced and investigated three interesting subclasses $\Sigma^*_{B,\lambda}(\alpha,\beta)$, $\Sigma^*_B(\lambda,\beta,\gamma)$ and $\widetilde{\Sigma}^*_{B,\lambda}(\beta,\gamma,\delta)$ of meromorphically bi-univalent functions defined on $\Delta = \{z \in \mathbb{C} : |z| > 1\}$ and established their initial coefficient estimates.

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1. Introduction

Let A be the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of A, which consists of functions of the form (1.1) which are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in U.

A function $f \in S$ is said to be starlike of order α $(0 \le \alpha < 1)$ in U if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$$

and is convex of order α $(0 \le \alpha < 1)$ in U if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha\,.$$

We denote these subclasses respectively by $S^*(\alpha)$ and $K(\alpha)$.
Also, a function $f \in S$ is said to be δ-spirallike of order γ $(0 \leq \gamma < 1)$ in U if

$$\Re\left(e^{i\delta} z \frac{f'(z)}{f(z)}\right) > \gamma cos\delta\,,$$

for some real δ such that $|\delta| < \dfrac{\pi}{2}$. The class of such functions is denoted by $S_p^{\gamma}(\delta)$.

It is well known that every function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}\left(f(z)\right) = z,\ \ (z \in U)$ and $f\left(f^{-1}(w)\right) = w,\ \left(|w| < r_0(f);\ \ r_0(f) \geq \dfrac{1}{4}\right)$.

A function $f \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions in U given by (1.1).

A systematic study of the class Σ was introduced in 1967 by Lewin [8], was revived in recent years by Srivastava *et al.*[10]. Ever since then, several authors investigated various subclasses of the class Σ and obtain estimates on the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.
(see, for example, [[3], [12], [13]]).

In our present investigation, the concept of bi-univalency is extended to the class of meromorphic functions defined on $\Delta = \{z \in \mathbb{C} : |z| > 1\}$.

The class of functions

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{1.2}$$

which are meromorphic and univalent in Δ and is denoted by Σ^*.

Since $g \in \Sigma^*$ is univalent, it has an inverse $g^{-1} = h$ that satisfies the following conditions:
$g^{-1}\left(g(z)\right) = z, \quad (z \in \Delta)$ and $g\left(g^{-1}(w)\right) = w, \qquad (0 < M < |w| < \infty)\,,$
where

$$g^{-1}(w) = h(w) = w + B_0 + \sum_{n=1}^{\infty} \frac{B_n}{w^n} \qquad (0 < M < |w| < \infty)\,. \tag{1.3}$$

A simple computation shows that

$$w = g(h(w)) = (b_0+B_0)+w+\frac{b_1+B_1}{w}+\frac{B_2-b_1B_0+b_2}{w^2}+\frac{B_3-b_1B_1+b_1B_0^2-2b_2B_0+b_3}{w^3}+\ldots \quad . \tag{1.4}$$

Comparing with initial coefficients in (1.4), we find that

$$b_0 + B_0 = 0 \implies B_0 = -b_0$$
$$b_1 + B_1 = 0 \implies B_1 = -b_1$$
$$B_2 - b_1B_0 + b_2 = 0 \implies B_2 = -(b_2 + b_0b_1)$$

$$B_3 - b_1B_1 + b_1B_0^2 - 2b_2B_0 + b_3 = 0 \implies B_3 = -(b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2)\,.$$

Equation (1.3) becomes,

$$g^{-1}(w) = h(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0b_1}{w^2} - \frac{b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2}{w^3} - \ \ldots\ . \quad (1.5)$$

Analogues to the bi-univalent analytic functions, a function $g \in \Sigma^*$ is said to be meromorphic bi-univalent function if $g^{-1} \in \Sigma^*$. The class of all meromorphic bi-univalent functions is denoted by Σ_B^*.

Estimates on the coefficients of meromorphically bi-univalent functions were widely investigated in the literature of Geometric function theory. Recently several researchers such as Halim *et al.*[5], Hamidi *et al.*[[6], [7]], Srivastava*et al.*[9] and Xiao *et al.*[11], introduced new subclasses of meromorphic bi-univalent functions and obtained estimates on the initial coefficients $|b_0|$ and $|b_1|$. Also in [1], Babalola defined and studied the class $\ell_\lambda(\beta)$ of λ-pseudo starlike functions of order β.

Motivated by the aforementioned work, in our present investigation, we introduce three new subclasses of the class Σ_B^* and obtained the estimates on the initial coefficients.

In order to derive our main results, we recall here the following Lemma.

Lemma 1. ([4], *see also* ([2], *p.*41)). *Let $p \in P$, where P is the family of all functions p, analytic in Δ for which $\Re\{p(z)\} > 0$ and have the form*
$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \ \ldots\ , (z \in \Delta)$. *Then $|p_n| \leq 2$ for each $n \in \mathbb{N}$.*

2. Coefficient bounds for the function class $\Sigma_{B,\lambda}^*(\alpha,\beta)$

We define the class $\Sigma_{B,\lambda}^*(\alpha,\beta)$ as follows:

Definition 1. *A function $g \in \Sigma_B^*$ given by (1.2) is said to be in the class $\Sigma_{B,\lambda}^*(\alpha,\beta)$ if the following conditions are satisfied :*

$$\left| arg\left(\frac{z[g'(z)]^\lambda}{(1-\beta)g(z) + \beta z g'(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \Delta) \qquad (2.1)$$

and

$$\left| arg\left(\frac{w[h'(w)]^\lambda}{(1-\beta)h(w) + \beta w h'(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \Delta), \qquad (2.2)$$

where $0 < \alpha \leq 1$, $0 \leq \beta < 1$, $\lambda \geq 1$ and the function h is the inverse of g given by (1.5).

We denote by $\Sigma^*_{B,\lambda}(\alpha,\beta)$, the class of functions which are meromorphic strongly λ-pseudo starlike bi-univalent of order α in Δ .

The estimates on the coefficients $|b_0|$ and $|b_1|$ for the class $\Sigma^*_{B,\lambda}(\alpha,\beta)$ are given as below.

Theorem 1. *Let g given by (1.2) be in the class $\Sigma^*_{B,\lambda}(\alpha,\beta)$. Then*

$$|b_0| \leq \frac{2\alpha}{1-\beta} \tag{2.3}$$

and

$$|b_1| \leq \frac{2\sqrt{5}\ \alpha^2}{1-2\beta+\lambda}\ . \tag{2.4}$$

Proof. Let $g \in \Sigma^*_{B,\lambda}(\alpha,\beta)$. Then by Definition 1, the conditions (2.1) and (2.2) can be rewritten as

$$\frac{z[g'(z)]^\lambda}{(1-\beta)g(z)+\beta z g'(z)} = [p(z)]^\alpha \tag{2.5}$$

and

$$\frac{w[h'(w)]^\lambda}{(1-\beta)h(w)+\beta w h'(w)} = [q(w)]^\alpha \tag{2.6}$$

respectively. Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \ \ldots \quad (z \in \Delta)$$

and

$$q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \frac{q_3}{w^3} + \ \ldots \quad (w \in \Delta)\ .$$

Clearly,

$$[p(z)]^\alpha = 1 + \frac{\alpha p_1}{z} + \frac{\alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2}{z^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)p_1^3 + \alpha(\alpha-1)p_1p_2 + \alpha p_3}{z^3} + \ldots$$

and

$$[q(w)]^\alpha = 1 + \frac{\alpha q_1}{w} + \frac{\alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2}{w^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)q_1^3 + \alpha(\alpha-1)q_1q_2 + \alpha q_3}{w^3} + \ldots\ .$$

Also,

$$\frac{z[g'(z)]^\lambda}{(1-\beta)g(z)+\beta zg'(z)} = 1 - \frac{(1-\beta)b_0}{z} + \frac{[(1-\beta)^2b_0^2 - (1-2\beta+\lambda)b_1]}{z^2} - \frac{[(1-\beta)^3b_0^3 - (1-\beta)(2-4\beta+\lambda)b_0b_1 + (1-3\beta+2\lambda)b_2]}{z^3} + \ \ldots$$

and

$$\frac{w[h'(w)]^\lambda}{(1-\beta)h(w)+\beta wh'(w)} = 1 + \frac{(1-\beta)b_0}{w} + \frac{[(1-\beta)^2b_0^2 + (1-2\beta+\lambda)b_1]}{w^2} + \frac{[(1-\beta)^3b_0^3 + (1+2\lambda+(1-\beta)(2-4\beta+\lambda))b_0b_1 + (1+3\beta+2\lambda)b_2]}{w^3} + \ \ldots \ .$$

Now, equating the coefficients in (2.5) and (2.6), we get

$$-(1-\beta)b_0 = \alpha p_1 \,, \tag{2.7}$$

$$(1-\beta)^2b_0^2 - (1-2\beta+\lambda)b_1 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 \,, \tag{2.8}$$

$$(1-\beta)b_0 = \alpha q_1 \,, \tag{2.9}$$

$$(1-\beta)^2b_0^2 + (1-2\beta+\lambda)b_1 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2 \,. \tag{2.10}$$

From equations (2.7) and (2.9), we get

$$p_1 = -q_1 \tag{2.11}$$

and

$$2(1-\beta)^2b_0^2 = \alpha^2(p_1^2+q_1^2) \,.$$

Using (2.11), we have

$$b_0^2 = \frac{\alpha^2p_1^2}{(1-\beta)^2} \,. \tag{2.12}$$

Applying Lemma 1, for the coefficient p_1 we have

$$|b_0|^2 \le \frac{4\alpha^2}{(1-\beta)^2} \quad \Rightarrow \ |b_0| \le \frac{2\alpha}{1-\beta} \,.$$

Which gives the bound on $|b_0|$ as given in (2.3).
Next, in order to find the bound on $|b_1|$, by using the equations (2.8) and (2.10), we

get

$$(1-\beta)^4 b_0^4 - (1-2\beta+\lambda)^2 b_1^2 = \frac{1}{4}\alpha^2(\alpha-1)^2 p_1^2 q_1^2 + \frac{1}{2}\alpha^2(\alpha-1)(p_1^2 q_2 + p_2 q_1^2) - \alpha^2 p_2 q_2 .$$

By simplifying and using (2.12), we have

$$(1-2\beta+\lambda)^2 b_1^2 = \alpha^4 p_1^4 - \frac{1}{4}\alpha^2(\alpha-1)^2 p_1^2 q_1^2 - \frac{1}{2}\alpha^2(\alpha-1)(p_1^2 q_2 + p_2 q_1^2) - \alpha^2 p_2 q_2 .$$

Applying Lemma 1, for the coefficients p_1, q_1, p_2 and q_2 we get

$$(1-2\beta+\lambda)^2 |b_1|^2 \leq 16\alpha^4 + 4\alpha^2(\alpha-1)^2 + 8\alpha^2(\alpha-1) + 4\alpha^2.$$

$$|b_1|^2 \leq \frac{20\alpha^4}{(1-2\beta+\lambda)^2} ,$$

$$\Rightarrow |b_1| \leq \frac{2\sqrt{5}\ \alpha^2}{1-2\beta+\lambda} .$$

Which gives the bound on $|b_1|$ as given in (2.4).
This completes the proof of Theorem 1.

3. Coefficient bounds for the function class $\Sigma_B^*(\lambda, \beta, \gamma)$

The definition of the class $\Sigma_B^*(\lambda, \beta, \gamma)$ is as follows:

Definition 2. *A function $g \in \Sigma_B^*$ given by* (1.2) *is said to be in the class $\Sigma_B^*(\lambda, \beta, \gamma)$ if the following conditions are satisfied :*

$$\Re\left(\frac{z[g'(z)]^\lambda}{(1-\beta)g(z) + \beta z g'(z)}\right) > \gamma \qquad (z \in \Delta) \tag{3.1}$$

and

$$\Re\left(\frac{w[h'(w)]^\lambda}{(1-\beta)h(w) + \beta w h'(w)}\right) > \gamma \qquad (w \in \Delta), \tag{3.2}$$

where $0 \leq \beta, \gamma < 1$, $\lambda \geq 1$ and the function h is the inverse of g given by (1.5).

We denote $\Sigma_B^*(\lambda, \beta, \gamma)$ the class of meromorphically λ-pseudo starlike bi-univalent function of order γ .

We now derive the estimates on the coefficients $|b_0|$ and $|b_1|$ for the meromorphically bi-univalent function class $\Sigma_B^*(\lambda, \beta, \gamma)$.

Theorem 2. *Let g given by (1.2) be in the class $\Sigma_B^*(\lambda,\beta,\gamma)$. Then*

$$|b_0| \leq \frac{2(1-\gamma)}{1-\beta} \tag{3.3}$$

and

$$|b_1| \leq \frac{2(1-\gamma)\sqrt{4\gamma^2-8\gamma+5}}{1-2\beta+\lambda}. \tag{3.4}$$

Proof. Let $g \in \Sigma_B^*(\lambda,\beta,\gamma)$. Then by Definition 2, the conditions (3.1) and (3.2) can be rewritten as follows:

$$\frac{z[g'(z)]^\lambda}{(1-\beta)g(z)+\beta z g'(z)} = \gamma + (1-\gamma)p(z) \tag{3.5}$$

and

$$\frac{w[h'(w)]^\lambda}{(1-\beta)h(w)+\beta w h'(w)} = \gamma + (1-\gamma)q(w) \tag{3.6}$$

respectively. Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \ \ldots \qquad (z \in \Delta)$$

and

$$q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \frac{q_3}{w^3} + \ \ldots \qquad (w \in \Delta)\ .$$

Clearly,

$$\gamma + (1-\gamma)p(z) = 1 + \frac{(1-\gamma)p_1}{z} + \frac{(1-\gamma)p_2}{z^2} + \frac{(1-\gamma)p_3}{z^3} + \ \ldots$$

and

$$\gamma + (1-\gamma)q(w) = 1 + \frac{(1-\gamma)q_1}{w} + \frac{(1-\gamma)q_2}{w^2} + \frac{(1-\gamma)q_3}{w^3} + \ \ldots \ .$$

Also,

$$\frac{z[g'(z)]^\lambda}{(1-\beta)g(z)+\beta z g'(z)} = 1 - \frac{(1-\beta)b_0}{z} + \frac{[(1-\beta)^2 b_0^2 - (1-2\beta+\lambda)b_1]}{z^2}$$
$$- \frac{[(1-\beta)^3 b_0^3 - (1-\beta)(2-4\beta+\lambda)b_0 b_1 + (1-3\beta+2\lambda)b_2]}{z^3} + \ \ldots$$

and

$$\frac{w[h'(w)]^{\lambda}}{(1-\beta)h(w)+\beta wh'(w)} = 1 + \frac{(1-\beta)b_0}{w} + \frac{[(1-\beta)^2 b_0^2 + (1-2\beta+\lambda)b_1]}{w^2}$$
$$+ \frac{[(1-\beta)^3 b_0^3 + (1+2\lambda+(1-\beta)(2-4\beta+\lambda))b_0 b_1 + (1+3\beta+2\lambda)b_2]}{w^3} + \ \dots \ .$$

Now, equating the coefficients in (3.5) and (3.6), we get

$$-(1-\beta)b_0 = (1-\gamma)p_1\,, \tag{3.7}$$

$$(1-\beta)^2 b_0^2 - (1-2\beta+\lambda)b_1 = (1-\gamma)p_2\,, \tag{3.8}$$

$$(1-\beta)b_0 = (1-\gamma)q_1\,, \tag{3.9}$$

$$(1-\beta)^2 b_0^2 + (1-2\beta+\lambda)b_1 = (1-\gamma)q_2\,. \tag{3.10}$$

From equations (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2(1-\beta)^2 b_0^2 = (1-\gamma)^2(p_1^2+q_1^2)\ ,$$

Using (3.11), we have

$$b_0^2 = \frac{(1-\gamma)^2 p_1^2}{(1-\beta)^2}\ . \tag{3.12}$$

Applying Lemma 1 for the coefficient p_1, we have

$$|b_0|^2 \le \frac{4(1-\gamma)^2}{(1-\beta)^2} \quad \Rightarrow \quad |b_0| \le \frac{2(1-\gamma)}{1-\beta}\ .$$

Which is the bound on $|b_0|$ as given in (3.3).
Next, in order to find the bound on $|b_1|$, by using the equations (3.8) and (3.10), we get

$$(1-\beta)^4\, b_0^4 - (1-2\beta+\lambda)^2\, b_1^2 = (1-\gamma)^2\, p_2 q_2\ .$$

By simplifying and using (3.12), we have

$$(1-2\beta+\lambda)^2\, b_1^2 = (1-\gamma)^4\, p_1^4 - (1-\gamma)^2\, p_2 q_2.$$

Applying Lemma 1 for the coefficients p_1, p_2 and q_2 we get

$$(1-2\beta+\lambda)^2\,|b_1|^2 \leq 16(1-\gamma)^4 + 4(1-\gamma)^2.$$

$$|b_1|^2 \leq \frac{4(1-\gamma)^2\,[4\gamma^2-8\gamma+5]}{(1-2\beta+\lambda)^2}\,,$$

$$\Rightarrow |b_1| \leq \frac{2(1-\gamma)\,\sqrt{4\gamma^2-8\gamma+5}}{1-2\beta+\lambda}\,.$$

Which gives the bound on $|b_1|$ as given in (3.4). This completes the proof of Theorem 2.

4. Coefficient bounds for the function class $\widetilde{\Sigma}^*_{B,\lambda}(\beta,\gamma,\delta)$

For the function g given by (1.2) with $b_1 = b_2 = ... = b_{k-1} = 0$, some estimates on the initial coefficients can be obtained. We define the class $\widetilde{\Sigma}^*_{B,\lambda}(\beta,\gamma,\delta)$ as follows:

Definition 3. *A function*

$$g(z) = z + b_0 + \sum_{n=k}^{\infty} \frac{b_n}{z^n} \tag{4.1}$$

is said to be in the class $\widetilde{\Sigma}^*_{B,\lambda}(\beta,\gamma,\delta)$ *where* $0 \leq \beta, \gamma < 1$, $\lambda \geq 1$ *and* $|\delta| < \frac{\pi}{2}$, *if the following conditions are satisfied:*

$$\Re\left(\frac{e^{i\delta} z[g'(z)]^{\lambda}}{(1-\beta)g(z)+\beta z g'(z)}\right) > \gamma cos\delta \qquad (z \in \Delta) \tag{4.2}$$

and

$$\Re\left(\frac{e^{i\delta} w[h'(w)]^{\lambda}}{(1-\beta)h(w)+\beta w h'(w)}\right) > \gamma cos\delta \qquad (w \in \Delta), \tag{4.3}$$

where the function h is the inverse of g given by

$$h(w) = w - b_0 - \frac{b_k}{w^k} - \frac{k b_0 b_k + b_{k+1}}{w^{k+1}} - \;\ldots\;. \tag{4.4}$$

We call $\widetilde{\Sigma}^*_{B,\lambda}(\beta,\gamma,\delta)$ the class of weakerly meromorphic λ-pseudo δ-spiralike bi-univalent functions of order γ .

We now derive the estimates on the coefficients for the function class $\widetilde{\Sigma}^*_{B,\lambda}(\beta,\gamma,\delta)$, we find the following result.

Theorem 3. *Let g given by (4.1) be in the class $\widetilde{\Sigma}^*_{B,\lambda}(\beta,\gamma,\delta)$. Then*

$$|b_0| \leq \frac{\left[4(1+\gamma(\gamma-2)cos^2\delta)\right]^{\frac{1}{2k}}}{1-\beta} \tag{4.5}$$

and
(a) for each positive odd integer k,

$$|b_k| \leq \frac{2\sqrt{[1+\gamma(\gamma-2)cos^2\delta]\left[1+4^{\frac{1}{k}}(1+\gamma(\gamma-2)cos^2\delta)^{\frac{1}{k}}\right]}}{1+\lambda k-\beta(1+k)}, \tag{4.6}$$

(b) for each positive even integer k,

$$|b_k| \leq \frac{2\sqrt{1+\gamma(\gamma-2)cos^2\delta}\left[1+2^{\frac{1}{k}}(1+\gamma(\gamma-2)cos^2\delta)^{\frac{1}{2k}}\right]}{1+\lambda k-\beta(1+k)}. \tag{4.7}$$

Proof. Let $g(z) = z + b_0 + \sum_{n=k}^{\infty} \frac{b_n}{z^n}$. Then by Definition 3, the conditions (4.2) and (4.3) can be rewritten as follows:

$$\frac{e^{i\delta}\, z\, [g'(z)]^{\lambda}}{(1-\beta)g(z)+\beta z g'(z)} = \gamma cos\delta + (e^{i\delta}-\gamma cos\delta)\, p(z) \tag{4.8}$$

and

$$\frac{e^{i\delta}\, w\, [h'(w)]^{\lambda}}{(1-\beta)h(w)+\beta w h'(w)} = \gamma cos\delta + (e^{i\delta}-\gamma cos\delta)\, q(w) \tag{4.9}$$

respectively. Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \ \ldots \qquad (z \in \Delta)$$

and

$$q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \frac{q_3}{w^3} + \ \ldots \qquad (w \in \Delta)\ .$$

Clearly,

$$\gamma cos\delta + (e^{i\delta}-\gamma cos\delta)p(z) = e^{i\delta} + \frac{(e^{i\delta}-\gamma cos\delta)p_1}{z} + \ldots + \frac{(e^{i\delta}-\gamma cos\delta)p_k}{z^k} + \frac{(e^{i\delta}-\gamma cos\delta)p_{k+1}}{z^{k+1}} + \ldots$$

and

$$\gamma cos\delta+(e^{i\delta}-\gamma cos\delta)q(w) = e^{i\delta}+\frac{(e^{i\delta}-\gamma cos\delta)q_1}{w}+\ldots+\frac{(e^{i\delta}-\gamma cos\delta)q_k}{w^k}+\frac{(e^{i\delta}-\gamma cos\delta)q_{k+1}}{w^{k+1}}+\ldots \quad .$$

Also,

$$\frac{e^{i\delta}z[g'(z)]^\lambda}{(1-\beta)g(z)+\beta zg'(z)} = e^{i\delta}-e^{i\delta}\frac{(1-\beta)b_0}{z}+\ \ldots+\frac{e^{i\delta}(-1)^k(1-\beta)^k b_0^k}{z^k} + \frac{e^{i\delta}[(-1)^{k+1}(1-\beta)^{k+1}b_0^{k+1}-(1+\lambda k-\beta(1+k))b_k]}{z^{k+1}}+\ \ldots$$

and using equation (4.4), we get

$$\frac{e^{i\delta}w[h'(w)]^\lambda}{(1-\beta)h(w)+\beta wh'(w)} = e^{i\delta}+e^{i\delta}\frac{(1-\beta)b_0}{w}+\ \ldots+\frac{e^{i\delta}(1-\beta)^k b_0^k}{w^k} + \frac{e^{i\delta}[(1-\beta)^{k+1}b_0^{k+1}+(1+\lambda k-\beta(1+k))b_k]}{w^{k+1}}+\ \ldots \quad .$$

Now, equating the coefficients in (4.8) and (4.9), we get

$$e^{i\delta}(-1)^k(1-\beta)^k b_0^k = (e^{i\delta}-\gamma cos\delta)\,p_k\,, \tag{4.10}$$

$$e^{i\delta}\left[(-1)^{k+1}(1-\beta)^{k+1}b_0^{k+1}-(1+\lambda k-\beta(1+k))\,b_k\right] = (e^{i\delta}-\gamma cos\delta)\,p_{k+1}\,, \tag{4.11}$$

$$e^{i\delta}(1-\beta)^k b_0^k = (e^{i\delta}-\gamma cos\delta)\,q_k\,,$$

$$e^{i\delta}\left[(1-\beta)^{k+1}b_0^{k+1}+(1+\lambda k-\beta(1+k))\,b_k\right] = (e^{i\delta}-\gamma cos\delta)\,q_{k+1}\,. \tag{4.12}$$

From equations (4.10), we get

$$b_0^k = \frac{(e^{i\delta}-\gamma cos\delta)p_k}{e^{i\delta}(-1)^k(1-\beta)^k}\,.$$

Using Lemma 1, we get

$$|b_0|^k \le \frac{2|(e^{i\delta}-\gamma cos\delta)|}{(1-\beta)^k}\,,$$

$$|b_0| \le \frac{\left[4(1+\gamma(\gamma-2)cos^2\delta)\right]^{\frac{1}{2k}}}{1-\beta}\,.$$

Which is the bound on $|b_0|$, as asserted in (4.5).

Next, in order to find the bound on $|b_k|$, for each positive odd integer k, multiplying both sides of (4.11) by both sides of (4.12), respectively we get

$$e^{2i\delta}\left[(1-\beta)^{2k+2}b_0^{2k+2}-(1+\lambda k-\beta(1+k))^2\,b_k^2\right]=(e^{i\delta}-\gamma cos\delta)^2 p_{k+1}\,q_{k+1}\ ,$$

$$[1+\lambda k-\beta(1+k)]^2\,b_k^2=-\frac{(e^{i\delta}-\gamma cos\delta)^2 p_{k+1}\,q_{k+1}}{e^{2i\delta}}+(1-\beta)^{2k+2}\,b_0^{2k+2}.$$

By using Lemma 1 and considering the bound on $|b_0|$, we conclude that

$$|b_k|\leq\frac{2\sqrt{[1+\gamma(\gamma-2)cos^2\delta]\left[1+4^{\frac{1}{k}}(1+\gamma(\gamma-2)cos^2\delta)^{\frac{1}{k}}\right]}}{1+\lambda k-\beta(1+k)}\,. \tag{4.13}$$

On the other hand, for every positive even integer k, from (4.12) and using the Lemma 1 and also considering the bound on $|b_0|$, we conclude that

$$|b_k|\leq\frac{2\sqrt{1+\gamma(\gamma-2)cos^2\delta}\left[1+2^{\frac{1}{k}}\left(1+\gamma(\gamma-2)cos^2\delta\right)^{\frac{1}{2k}}\right]}{1+\lambda k-\beta(1+k)}\,. \tag{4.14}$$

Equations (4.13) and (4.14) gives the bound on $|b_k|$ as asserted in (4.6) and (4.7) respectively. Hence, complete the proof of Theorem 3.

Remark 1. *By suitably specializing the various parameters involved in the assertion of Theorem 1, Theorem 2 and Theorem 3, we can deduce the corresponding coefficient estimates for several simpler meromorphically bi-univalent function classes.*

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